The Moduli Space of Hyperbolic Surfaces, Analytic Teichmüller Theory, and the Pants Graph

by

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Abstract: In the first section of this thesis we discuss fundamental topological properties of the Moduli space of hyperbolic surfaces. The next third is concerned with Teichmüller theory from the point of view of complex geometry in order to introduce the Weil-Petersson metric. In the last chapter we give an application of the rich structure of the Weil-Petersson metric by showing that it is quasi-isometric to the pants graph.

Contents

Introduction

Arguably, one of the most important results in two-dimensional geometry is the discovery that any compact surface has an underlying topological structure that depends only on two invariants of the surface, namely the genus and the number of boundary components. Therefore, the topological classification in dimension two is easily visualized. Another major result was that any compact surface for which twice the genus plus the number of boundary components is greater than 2 admits a hyperbolic structure. The goal to classify all such hyperbolic structures lead to the notion of the *Teichmüller space of hyperbolic surfaces*. The Teichmüller space of hyperbolic surfaces of genus g is the set of hyperbolic metrics on the unique topological surface of genus g modulo the action of orientation-preserving homeomorphisms isotopic to the identity, which act by pullback of metrics. The Teichmüller space can be visualized by an intuitively natural homeomorphism to standard euclidean space, the *Fenchel-Nielsen coordinates*. Given that the space of orientation-preserving homeomorphisms has a rich structure of connected components, one can also ask about the quotient of the space of hyperbolic metrics by the action of these connected components. The resulting space is called *Moduli space of hyperbolic surfaces*.

The content of this thesis is two-fold. The first goal is to understand the topology of Moduli space. This is done in chapter 1, where we prove *Fricke's theorem*, *Mumford's compactness criterion* and that Moduli space is simply connected, among other results. The second part of the thesis is concerned with Teichmüller theory from the point of view of complex geometry. Originally, this part was meant to be a preliminary section for the third chapter, in which we need the *Weil-Petersson metric*, but during the writing process it turned into an expansive chapter of its own. This second part introduces the Teichmüller space of a Fuchsian group and discusses all the intermediate steps needed to understand the well-known identification of the tangent and cotangent bundle of Teichmüller space with the space of *harmonic Beltrami differentials* and the space of *holomorphic quadratic differentials*, respectively. The chapter is concluded with constructing the Weil-Petersson metric and mentioning some results about its rich structure. Finally, the content of chapter 3 is the proof that the pants graph is quasi-isometric to Teichmüller space equipped with the Weil-Petersson metric. The pants graph is the graph that has pants decompositions of the genus g surface as vertices and has an edge between two vertices if they can be linked by an elementary move. The second and third chapter of the thesis can be read (almost) independently of the first.

Before starting, we would like to point out the method for referencing sources that we used. Each chapter has a main reference. If a result or a proof can be found in the main reference for its chapter, then it will not be explicitly stated. When a proof is taken from another source than the main one, it will be explicitly referenced. The main references are [6, p. 263-364] for chapter 1, [9, p. 77-253] for chapters 1.1, 1.4 and 2, and [4, p. 495-507] for chapter 3. The former two sources as well as [5] and [11] are a good starting point for getting accustomed to the background material. All figures are courtesy of the author, drawn with the Ipe extensible drawing editor.

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Notation

Let us begin by fixing some notation. $\mathcal{S}_{g,b}$ denotes the unique topological surface of genus g with b open disks removed, and $\mathcal{S}_g$ is the surface $\mathcal{S}_{g,b}$ with $b = 0$. Non-closed geodesics on a surface will be referred to as geodesic arcs. Moreover, if at some point we deal with geodesics that are not necessarily simple, then this will be emphasized. Generally, when we speak about geodesics, they are assumed to be closed and simple. The set of orientation-preserving homeomorphisms $\mathcal{S}_{g,b} \to \mathcal{S}_{g,b}$ that fix the boundary point-wise will be denoted by $\mathrm{Hom}^+(\mathcal{S}_{g,b})$ and the subset of homeomorphisms that are isotopic to the identity by

$\mathrm{Hom}_0(\mathcal{S}_{g,b})$. The quotient

$$\mathrm{MCG}(\mathcal{S}_{g,b}) = \mathrm{Hom}^+(\mathcal{S}_{g,b})/\mathrm{Hom}_0(\mathcal{S}_{g,b})$$

is the *Mapping Class Group* of the surface. A standing assumption throughout the entire thesis is that the surfaces in question always have negative Euler characteristic, i.e. $2 - 2g - b < 0$. Given this assumption, any surface admits pants decompositions. Slightly non-standard, we define a pants decomposition as a maximal collection of essential disjoint geodesics. Usually, the definition does not use geodesics but isotopy classes of simple closed curves. However, this is admissible since every isotopy class has a unique geodesic representative (cf. theorem A.2) so that our definition only has an impact on notation. Generally, recall that, for two-dimensional compact surfaces, isotopy and homotopy are interchangeable notions, and any homeomorphism is isotopic to a diffeomorphism. Lastly, note that if we call a map a Möbius transformation, then we refer to an element of $\mathrm{PSL}(2, \mathbb{R})$, not $\mathrm{PGL}(2, \mathbb{C})$. Recall that $\mathrm{PSL}(2, \mathbb{R})$ is exactly the automorphism group of the upper half plane $\mathbb{H} \subset \mathbb{C}$, which is the universal cover of any compact hyperbolic surface.

1 The Moduli Space of Hyperbolic Surfaces

1.1 Review of Teichmüller Theory

In this chapter, we recall the construction and some properties of the Teichmüller space of a surface. A hyperbolic metric is a Riemannian metric of constant curvature -1. A *hyperbolic structure* on $\mathcal{S}_{g,b}$ is a tuple (X, ϕ), where X is a surface with totally geodesic boundary and with a complete hyperbolic metric and $\phi\colon \mathcal{S}_{g,b} \to X$ is an orientation-preserving homeomorphism. We call the homeomorphism ϕ the *marking* of the *marked hyperbolic surface* (X, ϕ). Two hyperbolic structures (X_1, ϕ_1) and (X_2, ϕ_2) are said to be *homotopic* if there is an isometry $I\colon X_1 \to X_2$ such that the maps $I \circ \phi_1$ and ϕ_2 are homotopic in the usual sense or, equivalently, if $\phi_1 \circ \phi_2^{-1}$ is isotopic to an isometry. We define the *Teichmüller space* of $\mathcal{S}_{g,b}$ to be set of hyperbolic structures on $\mathcal{S}_{g,b}$ up to equivalence by homotopy,

$$\mathrm{Teich}(\mathcal{S}_{g,b}) = \big\{(X, \phi) \text{ hyperbolic structure on } \mathcal{S}_{g,b}\big\}/\text{homotopy}.$$

We often neglect the marking and denote an element of $\mathrm{Teich}(\mathcal{S}_{g,b})$ as $[X]$ instead of $[(X, \phi)]$. We can introduce a different view on Teichmüller space. Given a hyperbolic structure (X, ϕ), the pullback of the hyperbolic metric on X via ϕ gives a hyperbolic metric on $\mathcal{S}_{g,b}$. Since the notion of homotopy of two hyperbolic structures involves an isometry, any two homotopic markings induce the same hyperbolic metric on $\mathcal{S}_{g,b}$ up to changing the the metric by pullback by an element in $\mathrm{Hom}_0(\mathcal{S}_{g,b})$. Conversely, a hyperbolic metric d on $\mathcal{S}_{g,b}$ clearly gives us an element of $\mathrm{Teich}(\mathcal{S}_{g,b})$, namely $[(\mathcal{S}_{g,b}, d), id_{\mathcal{S}_{g,b}}]$. Moreover, for any hyperbolic metric d and any $f \in \mathrm{Hom}_0(\mathcal{S}_{g,b})$, the two hyperbolic structures $[(\mathcal{S}_{g,b}, d), id_{\mathcal{S}_{g,b}}]$ and $[(\mathcal{S}_{g,b}, f_*d), id_{\mathcal{S}_{g,b}}]$ are homotopic in the sense of markings because $[(\mathcal{S}_{g,b}, f_*d), id_{\mathcal{S}_{g,b}}]$ is the same as $[(\mathcal{S}_{g,b}, d), f^{-1}]$. Since the only isometry that is isotopic to the identity is the identity itself (see [14, p. 695]), the assignment

$$d \ (\mathrm{mod}\ \mathrm{Hom}_0(\mathcal{S}_{g,b})) \mapsto [(\mathcal{S}_{g,b}, d), id_{\mathcal{S}_{g,b}}]$$

is injective. The above map $[(X, d), \phi] \mapsto \phi_*d$ is a left inverse to this assignment. Thus, we have a bijection and may regard the Teichmüller space as the quotient of the set of hyperbolic metrics $\mathrm{HypM}(\mathcal{S}_{g,b})$ on $\mathcal{S}_{g,b}$ by the action of $\mathrm{Hom}_0(\mathcal{S}_{g,b})$ by pullback,

$$\mathrm{Teich}(\mathcal{S}_{g,b}) = \mathrm{HypM}(\mathcal{S}_{g,b})/\mathrm{Hom}_0(\mathcal{S}_{g,b}).$$

Despite the fact that an element of $\mathrm{Teich}(\mathcal{S}_{g,b})$ is an equivalence class of hyperbolic metrics, we can introduce a length function for each element in $\mathrm{Teich}(\mathcal{S}_{g,b})$. Given a point $[d] \in \mathrm{Teich}(\mathcal{S}_{g,b})$ and a representative d of this equivalence class, the length function $L_{[d]}$ sends an isotopy class $[c]$ of a simple closed curve c in $\mathcal{S}_{g,b}$ to the length of the geodesic representative of this isotopy class with respect to the hyperbolic metric d. Even though we chose a representative d of $[d]$, this is well-defined exactly because $L_{[d]}$ is defined on the set of isotopy classes of simple closed curves. Alternatively, we could also consider $L_{[d]}$ to take geodesics as input instead of allowing any simple closed curves. Hence, we can also write $L_{[d]}(\gamma)$ instead of $L_{[d]}([\gamma])$ whenever γ is a geodesic. Using the first view on $\mathrm{Teich}(\mathcal{S}_{g,b})$, the equivalence class $[d]$ corresponds to the equivalence class of a marked hyperbolic surface $[(X, \phi)]$. The length function $L_{[X]}$ sends an isotopy class of a simple closed curve c in $\mathcal{S}_{g,b}$ to the length of the geodesic representative of $\phi(c)$ in X.

Remark 1.1. *If L_X denotes the length function of the hyperbolic metric on X, then for any simple closed curve c in $\mathcal{S}_{g,b}$ we have $L_{[X]}([c]) = L_X([\phi(c)])$. Conversely, for a simple closed curve c in X we have $L_X([c]) = L_{[X]}([\phi^{-1}(c)])$. Beware the notation that L_X takes as input an isotopy class of a curve in X whereas $L_{[X]}$ takes as input an isotopy class of a curve in $\mathcal{S}_{g,b}$. In contrast to $L_{[X]}$, the length function L_X is not only defined for isotopy classes but for any curves. Distinguishing between c and $[c]$ as input, it should be clear from the context whether L_X is the actual length or the length of the geodesic representative.*

Let us now give $\mathrm{Teich}(\mathcal{S}_{g,b})$ a topology. Fix a pants decomposition $(\gamma_1,\ldots,\gamma_{3g-3+b})$ of $\mathcal{S}_{g,b}$ and let $\beta_1,\ldots,\beta_b$ denote the boundary components of $\mathcal{S}_{g,b}$. The *Fenchel-Nielsen coordinates* (short *FN* coordinates) consist of the $3g-3+2b$ *length coordinates*

$$(L_{[X]}(\gamma_1),\ldots,L_{[X]}(\gamma_{3g-3+b}),L_{[X]}(\beta_1),\ldots,L_{[X]}(\beta_b))$$

and the $3g-3+b$ *twist parameters*

$$(\theta_1([X]),\ldots,\theta_{3g-3}([X])),$$

where each $\theta_k([X])$ is the (signed) twist with which the two pairs of pants are attached to each other, normalized by the length $L_{[X]}(\gamma_k)$ of the gluing curve. It is an essential theorem that this defines a bijection

$$\begin{gathered} FN\colon \mathrm{Teich}(\mathcal{S}_{g,b}) \to \mathbb{R}_{>0}^{3g-3+2b}\times\mathbb{R}^{3g-3+b} \\ [X]\mapsto \big(L_{[X]}(\gamma_1),\ldots,L_{[X]}(\gamma_{3g-3+b}),L_{[X]}(\beta_1),\ldots,L_{[X]}(\beta_b),\theta_1([X]),\ldots,\theta_{3g-3+b}([X])\big), \end{gathered}$$

We give $\mathrm{Teich}(\mathcal{S}_{g,b})$ a topology by declaring this map to be a homeomorphism. Fortunately, this topology does not depend on the choice of Fenchel-Nielsen coordinates, as can be deduced from theorem 1.3 below. Interestingly, we do not need the twist parameters to specify elements in the Teichmüller space. Instead of taking $6g-6+3b$ length and twist coordinates we can also take $9g-9+3b$ length coordinates as the following theorem shows.

Theorem 1.2. *There are $9g-9+3b$ simple closed curves $\alpha_1,\ldots,\alpha_{9g-9+3b}$ in $\mathcal{S}_{g,b}$ such that*

$$\mathrm{Teich}(\mathcal{S}_{g,b})\to\mathbb{R}_{>0}^{9g-9+3b},\ [X]\mapsto\big(L_{[X]}([\alpha_1]),\ldots,L_{[X]}([\alpha_{9g-9+3b}])\big)$$

is injective.

Next, we want to give Teichmüller space a metric. We will later give a rigorous discussion of analytic Teichmüller theory, but at this point we already need some notions from that theory. Let us restrict to the case of closed surfaces of type $\mathcal{S}_g$, $g\geq 2$. Recall that any hyperbolic surface arises as the quotient of the upper half plane $\mathbb{H}$ by some group of Möbius transformations as do Riemann surfaces. Thus, we have a natural transition between the setting of hyperbolic surfaces and the setting of Riemann surfaces. Any orientation-preserving homeomorphism automatically is a quasi-conformal map since we are dealing with compact Riemann surfaces so that the markings correspond to quasi-conformal maps. Moreover, since any isometry between hyperbolic surfaces lifts to a Möbius transformation on $\mathbb{H}$ just as any conformal map between Riemann surfaces does, isometries in the hyperbolic setting correspond to conformal maps in the complex setting. There is a minor difference, though. In order to speak of quasi-conformal maps, we need to have the domain surface and the image surface equipped with a complex structure. Thus, it does not make sense to speak of $\mathrm{Teich}(\mathcal{S}_g)$ in the complex setting. Rather, we have to fix a Riemann surface $\mathcal{S}$ homeomorphic to $\mathcal{S}_g$ and consider the analogously defined $\mathrm{Teich}(\mathcal{S})$, which is bijective to $\mathrm{Teich}(\mathcal{S}_g)$. We prove later that the choice of $\mathcal{S}$ is unambiguous. Therefore, we can work with the following Teichmüller space in the complex setting:

$$\mathrm{Teich}(\mathcal{S})=\big\{[(X,\phi)]\ \big|\ \phi\colon\mathcal{S}\to X \text{ is quasi-conformal, where } X \text{ is some Riemann surface}\big\}\,,$$

where the equivalence class $[(X,\phi)]$ is defined by $(X,\phi)\sim(Y,\psi)$ if $\psi\circ\phi^{-1}$ is isotopic to a conformal map. We also neglect the map ϕ in the notation and often simply write $[X]$. Next, given any two points $[(X,\phi)],[(Y,\psi)]\in\mathrm{Teich}(\mathcal{S})$, we let $\mathcal{F}$ denote the set of quasi-conformal maps $X\to Y$ that are isotopic to $\psi\circ\phi^{-1}$. We call $\psi\circ\phi^{-1}$ the *change-of-marking map*. We define

$$\mathrm{d}_{\mathrm{Teich}}([X],[Y])=\inf_{h\in\mathcal{F}}\log(K(h))/2,$$

where $K(h)$ is the maximal dilatation of h.

Theorem 1.3. $\mathrm{d}_{\mathrm{Teich}}$ *is a complete metric on* $\mathrm{Teich}(\mathcal{S})$ *and induces the same topology as any Fenchel-Nielsen coordinates.*

We postpone the proof of this theorem to chapter 2.3 because it requires some notions and results that have not been developed yet. Teichmüller proved that the infimum in the definition of $\mathrm{d}_{\mathrm{Teich}}$ is actually realized by some map h and this map is uniquely determined by some specific property. To make this more precise, we introduce the notion of a holomorphic quadratic differential. Let X be a Riemann surface and denote its (complexified) tangent bundle by TX and its (complexified) cotangent bundle by T^*X. A (complex) basis for T^*X is given by $(dz, d\overline{z})$, and $(dz^2, dzd\overline{z}, d\overline{z}^2)$ is a (complex) basis for the second symmetric power of the cotangent bundle $S^2(T^*X)$. A *holomorphic quadratic differential* is a holomorphic section of the $\{dz^2\}$-span of $S^2(T^*X)$, i.e. is a map

$$q\colon X \to \{dz^2\}\text{-span of } S^2(T^*X),\ x \mapsto q_x \in \{(dz_x)^2\}\text{-span of } S^2(T_x^*X),$$

where the dependence of q_x on x is holomorphic. In particular, $q_x(v)$ is locally of the form $\phi(x)(dz_x)(v)^2$, $x \in X$, $v \in T_xX$, for some holomorphic function $\phi\colon U \to \mathbb{C}$ and some neighborhood U of x. By a usual change of coordinates argument, for any holomorphic quadratic differential q, there are some *natural coordinates* $z_\alpha\colon U_\alpha \to \mathbb{C}$, in which q takes the form $q_x(v) = z_\alpha(x)^k(dz_x)(v)^2$, $x \in U_\alpha$, $v \in T_xX$. The power k depends on the chart (U_α, z_α) and on q. In fact, k is the order of zero of $\phi(x_\alpha) = 0$, where x_α is the center of the coordinate chart U_α and ϕ is any local expression of q around x_α (with the convention $k = 0$ if $\phi(x_\alpha) \neq 0$). Indeed, this is independent of the local choice of ϕ: if $q_x = \phi(x)(dz_x)^2 = \psi(x)(dw_x)^2$ in two local coordinates z and w around x, then the order of zero of ϕ and ψ at x must agree since $\frac{dz_x}{dw_x}$ never vanishes. We define the norm of a holomorphic quadratic differential q to be $||q|| = 2\int_X q$. We can now state Teichmüller's existence and uniqueness theorem.

Theorem 1.4 (Teichmüller)**.** *Given a quasi-conformal map $f\colon X \to Y$ between two closed Riemann surfaces, there exists a unique quasi-conformal map $h\colon X \to Y$ that is isotopic to f and satisfies*[1]

$$\frac{\overline{\partial} h}{\partial h} = ||q|| \frac{\overline{q}}{|q|}$$

for some holomorphic quadratic differential q on X. Moreover, the dilatation of f is larger or equal to $||q||$ with equality if and only if $f = h$.

Thus, this so-called *Teichmüller map* is minimizing the dilatation in a given homotopy class. As an immediate consequence we obtain:

Corollary 1.5. *If h is the Teichmüller map isotopic to $\psi \circ \phi^{-1}$, $[(X,\phi)], [(Y,\psi)] \in \mathrm{Teich}(\mathcal{S})$, then* $\mathrm{d}_{\mathrm{Teich}}([X],[Y]) = \log(K(h))/2$.

Lastly, let us discuss that the choice of $\mathcal{S}$ is unambiguous. Fix a point $[(X', \phi')]$ in $\mathrm{Teich}(\mathcal{S})$. Then ϕ' induces a map

$$[\phi']_*\colon \mathrm{Teich}(\mathcal{S}) \to \mathrm{Teich}(X'),\ [(X,\phi)] \mapsto [(X, \phi \circ \phi'^{-1})].$$

Since ϕ'^{-1} induces a similar map from $\mathrm{Teich}(X')$ to $\mathrm{Teich}(\mathcal{S})$ that clearly is the inverse map to $[\phi']_*$, the latter must be bijective. Moreover, the set $\mathcal{F}$ of maps isotopic to $\psi \circ \phi^{-1}$ is the same as the set of maps isotopic to $\psi \circ \phi'^{-1} \circ (\phi \circ \phi')^{-1}$. This shows that $[\phi']_*$ is an isometry with respect to the Teichmüller metrics on $\mathrm{Teich}(\mathcal{S})$ and $\mathrm{Teich}(X')$. In particular, the identification $\mathrm{Teich}(\mathcal{S}_g) \simeq \mathrm{Teich}(\mathcal{S})$ gives us a well-defined Teichmüller metric on $\mathrm{Teich}(\mathcal{S}_g)$ independent of the choice of $\mathcal{S}$.

[1] This is the Beltrami coefficient of h. We will discuss Beltrami coefficients in chapter 2.

1.2 Construction of Moduli Space

Moduli space will be constructed as a quotient of Teichmüller space by a group action. Let us start by defining the latter. The group acting on $\mathrm{Teich}(\mathcal{S}_g)$ will be the Mapping Class Group. We can define an action of $\mathrm{MCG}(\mathcal{S}_g)$ on $\mathrm{Teich}(\mathcal{S}_g)$ as follows. Given $[f] \in \mathrm{MCG}(\mathcal{S}_g)$ and $[(X, \phi)] \in \mathrm{Teich}(\mathcal{S}_g)$, we choose a representative $f \in [f]$ and set

$$f \cdot [(X, \phi)] = [(X, \phi \circ f^{-1})].$$

Let us check that this is well-defined. Furthermore, let us also prove that this is an action by isometries, which will play a crucial role later on.

Proposition 1.6. *This is a well-defined group action. Using the definition of* $\mathrm{Teich}(\mathcal{S}_g)$ *with hyperbolic metrics, this action is the same as the action by pullback. Moreover, the action is isometric.*

Proof. If f' is another representative of $[f]$, then $f^{-1} \circ f'$ is isotopic to the identity and, hence, $\phi \circ f^{-1}$ and $\phi \circ f'^{-1}$ are homotopic in the sense of markings. This proves that the action is well-defined. If $f \in \mathrm{Hom}_0(\mathcal{S}_g)$ acts on d by pullback, then the resulting metric $f_* d$ gets mapped to $[(\mathcal{S}_g, f_* d), id_{\mathcal{S}_g}]$ by the bijection discussed last chapter, which is homotopic to $[(\mathcal{S}_g, d), f^{-1}]$ as markings. Given two markings $[(X, \phi)]$ and $[(Y, \psi)]$, the change-of-marking map is unchanged after replacing ϕ and ψ by $\phi \circ f^{-1}$ and $\psi \circ f^{-1}$, respectively. Hence, the set $\mathcal{F}$ in the definition of $\mathrm{d}_{\mathrm{Teich}}$ remains unchanged, and this shows that $\mathrm{MCG}(\mathcal{S}_g)$ acts isometrically on $\mathrm{Teich}(\mathcal{S}_g)$. $\square$

We now define the *Moduli space* of $\mathcal{S}_g$ as the quotient

$$\mathcal{M}(\mathcal{S}_g) = \mathrm{Teich}(\mathcal{S}_g)/\mathrm{MCG}(\mathcal{S}_g).$$

By the second statement of the last proposition, this quotient is the same as

$$\mathcal{M}(\mathcal{S}_g) = \mathrm{HypM}(\mathcal{S}_g)/\mathrm{Hom}^+(\mathcal{S}_g).$$

Note that the action of $\mathrm{MCG}(\mathcal{S}_g)$ on $\mathrm{Teich}(\mathcal{S}_g)$ only affects the marking ϕ of $[(X, \phi)]$. Thus, we can write an element of $\mathcal{M}(\mathcal{S}_g)$ as the hyperbolic surface X with all the properties of a hyperbolic surface, implicitly remembering that it is, in fact, an equivalence class of marked hyperbolic surfaces. Henceforth, we will simply denote an element of $\mathcal{M}(\mathcal{S}_g)$ by X and think of Moduli space as the space of hyperbolic surfaces homeomorphic to $\mathcal{S}_g$. Observe that $\mathcal{M}(\mathcal{S}_g)$ is the quotient of $\mathrm{Teich}(\mathcal{S}_g)$ by the action of the connected components of $\mathrm{Hom}^+(\mathcal{S}_g)$, as noted in the introduction. Thus, in the literature it is also common to write

$$\mathcal{M}(\mathcal{S}_g) = \mathrm{Teich}(\mathcal{S}_g)/\pi_0(\mathrm{Hom}^+(\mathcal{S}_g)).$$

The rest of this chapter will be concerned with proving that the action of $\mathrm{MCG}(\mathcal{S}_g)$ is properly discontinuous, known as Fricke's theorem.

Theorem 1.7 (Fricke)**.** *The action of* $\mathrm{MCG}(\mathcal{S}_g)$ *on* $\mathrm{Teich}(\mathcal{S}_g)$ *is properly discontinuous, i.e. for every compact subset* $K \subset \mathrm{Teich}(\mathcal{S}_g)$*, the set*

$$\left\{[f] \in \mathrm{MCG}(\mathcal{S}_g) \;\middle|\; [f] \cdot K \cap K \neq \emptyset\right\}$$

is finite.

We need some preliminary results first. Let us begin by investigating the *raw length spectrum* of a hyperbolic surface X, which is the set

$$\mathrm{RLS}(X) = \left\{L_X([c]) \;\middle|\; c \text{ is a simple closed curve in } X\right\}.$$

Since we have a well-defined length function for elements of $\mathrm{Teich}(\mathcal{S}_g)$, the same definition works for an equivalence class $[X] \in \mathrm{Teich}(\mathcal{S}_g)$,

$$\mathrm{RLS}([X]) = \left\{ L_{[X]}([c]) \mid c \text{ is a simple closed curve in } \mathcal{S}_g \right\}.$$

In our case of compact surfaces, the raw length spectrum will always be a discrete set. This is the content of the next lemma.

Lemma 1.8. *If X is a compact hyperbolic surface, then the set*

$$\left\{ [c] \;\middle|\; c \textit{ is a simple closed curve in } X \textit{ with } L_X([c]) < L \right\}$$

is finite, for any $L \geq 0$. In particular, $\mathrm{RLS}(X)$ *is a discrete set.*

Proof. Given $L > 0$, suppose γ is a geodesic in X of length at most L. Let $K \subset \mathbb{H}$ be a compact set containing the ball of radius L around a fundamental domain. This is possible because any fundamental domain of a compact surface is compact. Fix any $x_0 \in X$ as a base point for the fundamental group $\pi_1(X, x_0)$. Take $[\gamma'] \in \pi_1(X, x_0)$ so that γ' is homotopic to γ. Since we fixed a fundamental domain, we may need to conjugate $[\gamma']$ with another element in $\pi_1(X, x_0)$ such that we may assume that the action of $[\gamma']$ on the upper half plane satisfies $d_{\mathbb{H}}(x, [\gamma'] \cdot x) = L_X(\gamma) < L$, for points $x \in K$ on a lift of γ to K. Then $[\gamma'] \cdot K \cap K \neq \emptyset$. Since the action is properly discontinuous, only finitely many $[\gamma']$ can fulfill this property. Hence, only finitely many different γ were eligible in the first place. □

The second key ingredient we need for the proof of Fricke's theorem is a connection between the Teichmüller distance and the length functions.

Lemma 1.9 (Wolpert's lemma). *Let $\phi\colon X_1 \to X_2$ be a K-quasi-conformal homeomorphism between two hyperbolic surfaces X_1 and X_2. Then, for any simple closed curve c in X_1, we have*

$$\frac{L_{X_1}([c])}{K} \leq L_{X_2}([\phi(c)]) \leq K \cdot L_{X_1}([c]).$$

Proof. Suppose γ_1 and γ_2 are geodesics in X_1 and X_2, respectively, where γ_2 is isotopic to $\phi(\gamma_1)$. Let $\gamma_1' \subset \mathbb{H}$ be a geodesic in the hyperbolic plane obtained by concatenating lifts of γ_1. Take any point on γ_1 as a base point for the fundamental group of X_1. Then the action of $[\gamma_1]$ on $\mathbb{H}$ leaves γ_1' invariant and $d_{\mathbb{H}}(x, [\gamma_1] \cdot x) = L_{X_1}(\gamma_1)$, for all points $x \in \gamma_1'$. Let h_1 be a Möbius transformation that sends γ_1' to the imaginary axis, and suppose that $g_1 = h_1 \circ [\gamma_1] \circ h_1^{-1}$ is represented by $\frac{az+b}{cz+d}$. Since γ_1' is invariant under $[\gamma_1]$, the imaginary axis is invariant under g_1, which implies that we must have either $a = d = 0$ or $b = c = 0$. Which of these two cases occurs depends on whether $[\gamma_1]$ acts clockwise or counter-clockwise on γ_1'. By replacing γ_1 with the inversely parametrized curve in the second case, we can assume that we are always dealing with the former case. Thus, g_1 is of the form $z \mapsto \lambda z$ for some $\lambda > 0$. Then we compute

$$L_{X_1}(\gamma_1) = d_{\mathbb{H}}\left(h_1^{-1}(i), [\gamma_1](h_1^{-1}(i)\right) = d_{\mathbb{H}}(i, g_1(i)) = |\ln(\lambda)|.$$

Hence, we either have $\lambda = e^{L_{X_1}(\gamma_1)}$ or $\lambda = e^{-L_{X_1}(\gamma_1)}$. Let sign_1 denote the sign of the exponent so that $\lambda = e^{\mathrm{sign}_1 L_{X_1}(\gamma_1)}$. By the same argument, we may conjugate $[\gamma_2]$ with a Möbius transformation to a map g_2 of the form $z \mapsto e^{\mathrm{sign}_2 L_{X_2}(\gamma_2)} z$. Now consider the logarithm defined using the branch cut at the non-positive real axis. The upper half plane gets mapped to the strip $\{\Im(z) \in (0, \pi)\}$. The action of g_1 on $\ln(\mathbb{H}) \to \ln(\mathbb{H})$ is given by

$$z = \ln(w) \mapsto \ln(g_1(w)) = \ln(e^{\mathrm{sign}_1 L_{X_1}(\gamma_1)} w) = \mathrm{sign}_1 L_{X_1}(\gamma_1) + z,$$

i.e. it simply is horizontal translation by $L_{X_1}(\gamma_1)$. Therefore, a fundamental domain for the action is given by the rectangle $R_1 = \{\Re(z) \in [0, L_{X_1}(\gamma_1)),\ \Im(z) \in (0,\pi)\}$. Let C_1 be the cylinder obtained by identifying the vertical boundaries $\{\Re(z) = 0\}$ and $\{\Re(z) = L_{X_1}(\gamma_1)\}$. Since the logarithm is a conformal map, the quotient $A_1 = \mathbb{H}/\langle g_1\rangle$ is conformally equivalent to C_1. Hence, A_1 has the same modulus[2] as C_1, which simply is the height π divided by the width $L_{X_1}(\gamma_1)$. We can repeat the argument for g_2 to find that the quotient $A_2 = \mathbb{H}/\langle g_2\rangle$ has modulus $\pi/L_{X_2}(\gamma_2)$. Next, we consider the lift of ϕ to the universal cover $\mathbb{H}$, which descends to a map $\phi'\colon \mathbb{H}/\langle[\gamma_1]\rangle \to \mathbb{H}/\langle[\gamma_2]\rangle$. This is a K-quasi-conformal map. Since g_1 and g_2 are obtained from $[\gamma_1]$ and $[\gamma_2]$ by conjugation, we can conjugate ϕ' to a map $\phi_0 = h_2 \circ \phi' \circ h_1^{-1}\colon A_1 \to A_2$. Since h_2 and h_1^{-1} are Möbius transformations, ϕ_0 is also K-quasi-conformal. Lastly, we can use the conformal equivalence of A_i and C_i, $i = 1, 2$, to get a K-quasi-conformal map $\psi = \ln|_{A_2} \circ \phi_0 \circ (\ln|_{A_1})^{-1}\colon C_1 \to C_2$. Since this map is K-quasi-conformal, the modulus of image$(\psi) = C_2$ is some number in the interval $[\frac{\operatorname{mod}(C_1)}{K}, K\operatorname{mod}(C_1)]$ (see lemma 1.10 below). Using the explicit values for the moduli of C_1 and C_2, this reads

$$\frac{1}{K}\frac{\pi}{L_{X_1}(\gamma_1)} \le \frac{\pi}{L_{X_2}(\gamma_2)} \le K\frac{\pi}{L_{X_1}(\gamma_1)}.$$

Rearranging these inequalities yields the desired statement. □

In the last proof, we used the following lemma about the distortion of the modulus by a quasi-conformal map, called the solution to Grötzsch's problem for the special case of rectangles.

Lemma 1.10. *Suppose R_1 and R_2 are the two rectangles $[0, r_1] \times [0, s]$ and $[0, r_2] \times [0, s]$, respectively, and $f\colon R_1 \to R_2$ is a K-quasi-conformal map that takes vertical sides of the rectangle R_1 to vertical sides of R_2. Then the ratio r_2/r_1 is bounded by K. In other words, the ratio between the modulus of R_1 and the modulus of R_2 is bounded by K. Since f takes vertical sides of R_1 to vertical sides of R_2, the same holds for the quotient map between the annuli obtained by identifying vertical sides in each rectangle.*

Proof. The differential df is of the form $f_x dx + f_y dy$. Switching to complex notation, we can also regard df as $f_z dz + f_{\bar z} d\bar z$. The Beltrami coefficient of f at p is $\mu_f(p) = f_{\bar z}(p)/f_z(p)$. Consider the expressions

$$M(p) = |f_z(p)|(1 + |\mu_f(p)|),$$
$$m(p) = |f_z(p)|(1 - |\mu_f(p)|).$$

Then $M(p)/m(p)$ is the dilatation $K_f(p)$ of f at p, and a short calculation shows that $M(p)\cdot m(p)$ is exactly the determinant of $df(p)$. Another easy calculation reveals that we always have $M(p)^2 \ge |f_x(p)|^2$. Since f maps vertical sides to vertical sides, we have

$$r_2 \le \int_0^{r_1} |f_x(x, y)|\ dx,$$

for all $y \in [0, s]$ and, hence,

$$r_2 s \le \int_{R_1} |f_x(p)|\ dA.$$

We can combine these inequalities and use the Cauchy-Schwarz inequality for integrals to find out that

$$(r_2 s)^2 \le \left(\int_{R_1} \underbrace{|f_x(p)|}_{\le M(p)}\ dA\right)^2 \le \left(\int_{R_1} \sqrt{\frac{M(p)}{m(p)}}\sqrt{M(p)m(p)}\ dA\right)^2 =$$
$$= \left(\int_{R_1} \sqrt{K_f(p)}\sqrt{\det(df(p))}\ dA\right)^2 \le \left(\int_{R_1} K_f(p)\ dA\right)\cdot\left(\int_{R_1} \det(df(p))\ dA\right).$$

[2]For a quick introduction on moduli of cylinder and annuli, the reader may consult [13, p. 208ff.].

Since, by the transformation formula,

$$\int_{R_1} \det(df(p))\, dA = \int_{f(R_1)} dA = \text{Area}(f(R_1)),$$

and $K_f(p) \leq K$, we can compute further

$$(r_2 s)^2 \leq \left(\int_{R_1} K_f(p)\, dA\right) \cdot \left(\int_{R_1} \det(df(p))\, dA\right) \leq K \text{Area}(R_1) \cdot \text{Area}(R_2) = K r_1 s^2 r_2.$$

This proves that $r_2/r_1 \leq K$. Since we could have done the same with f^{-1}, we also have that $r_1/r_2 \leq K$, so that the ratio of the moduli of R_1 and R_2 is bounded by K. $\square$

In the proof of Fricke's theorem, we will not be using Wolpert's lemma directly but one of its immediate consequences for elements of Teichmüller space.

Corollary 1.11. *If* $[(X,\phi)], [(Y,\psi)] \in \text{Teich}(\mathcal{S}_g)$ *satisfy* $\mathrm{d}_{\text{Teich}}([X],[Y]) \leq \log(K)/2$*, then, for any simple closed curve* c *in* $\mathcal{S}_g$*,*

$$\frac{L_{[X]}([c])}{K} \leq L_{[Y]}([c]) \leq K \cdot L_{[X]}([c]).$$

Proof. Unraveling the definition of $\mathrm{d}_{\text{Teich}}$, the hypothesis state that we can find quasi-conformal maps $f_\epsilon \colon X \to Y$ in the isotopy class of the change of marking map $\psi \circ \phi^{-1}$ with dilatation at most $K+\epsilon$, where ϵ is arbitrarily small. The statement is now immediate from Wolpert's lemma:

$$\begin{aligned}\frac{L_{[X]}([c])}{K+\epsilon} = \frac{L_X([\phi(c)])}{K+\epsilon} &\leq L_Y([f_\epsilon \circ \phi(c)]) = L_Y([\psi(c)]) = \\ &= L_{[Y]}([c]) \leq (K+\epsilon) \cdot L_X([\phi(c)]) = (K+\epsilon) \cdot L_{[X]}([c]).\end{aligned}$$

$\square$

We can now prove Fricke's theorem. Recall that we want to prove proper discontinuity of the action of the Mapping Class Group on Teichmüller space.

Proof of theorem 1.7. Take a compact subset $K \subset \text{Teich}(\mathcal{S}_g)$, let D denote its diameter, and take any $[X] \in K$. Suppose $[f] \cdot K \cap K \neq \emptyset$, where $[f] \in \text{MCG}(\mathcal{S}_g)$. Since the action of $\text{MCG}(\mathcal{S}_g)$ is isometric, we have

$$\mathrm{d}_{\text{Teich}}([X], [f] \cdot [X]) \leq \text{diameter}(K \cup [f] \cdot K) \leq 2D.$$

For two simple, closed, essential curves c_1 and c_2 in $\mathcal{S}_g$, corollary 1.11 tells us that

$$L_{[f]\cdot[X]}([c_i]) \leq e^{4D} \cdot L_{[X]}([c_i]),\ i = 1,2.$$

Fix a representative f of $[f]$, and note that the following steps are all invariant under picking a different representative, since we only consider isotopy classes of curves. By remark 1.1, $L_{[f]\cdot[X]}([c_i]) = L_{[X]}([f^{-1}(c_i)])$, $i = 1,2$, and, hence,

$$[f^{-1}(c_i)] \in \left\{[c] \;\middle|\; L_X([c]) < (e^{4D}+1) \cdot L\right\},\ i = 1,2,$$

where $L = \max\{L_{[X]}([c_1]), L_{[X]}([c_2])\}$. Now, lemma 1.8 tells us that the latter is a finite set. Thus, $[f^{-1}(c_1)]$ and $[f^{-1}(c_2)]$ can only be two of finitely many isotopy classes. It is a standard result that

$\mathcal{S}_g$ can be filled by two curves[3] that are in minimal position and not isotopic to each other, so we might as well have chosen $[c_1]$ and $[c_2]$ to satisfy these properties. We claim that the set of mapping classes $[g]$ with $[g(c_i)] \in \{[c_1],[c_2]\}$ is finite. Indeed, the Alexander method (see [6, p. 59]) tells us that any homeomorphism g with the property $[g(c_i)] \in \{[c_1],[c_2]\}$ that also fixes the intersection points of $[c_1] \cap [c_2]$ and preserves the orientations of the curves is isotopic to the identity. Since there only finitely many permutations of how to map the intersection points to other intersections points and how to alter the orientations, the claim follows. Now suppose $g \in \mathrm{Hom}^+(\mathcal{S}_g)$, and $g^{-1}(c_1)$ and $g^{-1}(c_2)$ are in the isotopy classes $[f^{-1}(c_1)]$ and $[f^{-1}(c_2)]$, respectively. Any representative of $[f \circ g^{-1}]$ sends the isotopy classes $[c_1]$ and $[c_2]$ to themselves. Then the claim implies that $[f \circ g^{-1}]$ lies in a finite set, which depends only on c_1 and c_2. This means that once the isotopy classes $[f^{-1}(c_1)]$ and $[f^{-1}(c_2)]$ are determined, then there are only finitely many possibilities for the element $[f] \in \mathrm{MCG}(\mathcal{S}_g)$. We conclude that the set

$$\big\{[f] \in \mathrm{MCG}(\mathcal{S}_g) \;\big|\; [f] \cdot K \cap K \neq \emptyset\big\}$$

is finite. □

One corollary of Fricke's theorem is that we can metrize $\mathcal{M}(\mathcal{S}_g)$ with the Teichmüller metric, as the next proposition shows.

Proposition 1.12. *If (X,d) is a metric space and G is a group acting on X properly discontinuous by isometries, then d induces a metric on the quotient space.*

Proof. Consider the following map on $X/G \times X/G$:

$$d_{X/G}([x],[y]) = \inf_{g,h \in G} d(g \cdot x, h \cdot y).$$

This clearly is non-negative and satisfies $d_{X/G}([x],[y]) = 0$, whenever $[x] = [y]$. Since the group action is isometric, we can rewrite $d_{X/G}$ as

$$d_{X/G}([x],[y]) = \inf_{g,h \in G} d(g \cdot x, h \cdot y) = \inf_{g \in G} d(g \cdot x, y).$$

The triangle inequality follows easily:

$$\begin{aligned} d_{X/G}([x],[y]) &= \inf_{g \in G} d(g \cdot x, y) \leq \inf_{h \in H} \inf_{g \in G} \left(d(g \cdot x, h \cdot z) + d(h \cdot z, y)\right) = \\ &= \inf_{h \in H} \big(d_{X/G}([x],[z]) + d(h \cdot z, y)\big) = d_{X/G}([x],[z]) + d_{X/G}([z],[y]), \end{aligned}$$

where the equality at the line break uses the isometry of the action. So far, we did not use proper discontinuity of the action. Thus, if we drop that assumption, then d still induces a semi-metric on the quotient. To get an actual metric we need to show that $d_{X/G}([x],[y]) = 0$ implies $[x] = [y]$. To see this, we claim that the infimum in $\inf_{g \in G} d(g \cdot x, y)$ is actually a minimum. Define $D = d_{X/G}([x],[y])$ and suppose that there is a sequence $\{g_n\}_{n \geq 1} \subset G$ with $d(g_n \cdot x, y) \to D$ as $n \to \infty$. If K is the union of a compact ball around x and a compact ball of radius at least $2D$ around y, then $g_n \cdot K \cap K \neq \emptyset$, for all $n \geq N$ and some $N \geq 1$. By proper discontinuity of the action, the set $\{g_n\}_{n \geq 1}$ is finite, and the infimum is a minimum. Now that this is established, obviously $d_{X/G}([x],[y]) = 0$ implies $[x] = [y]$. □

Note that, aside from proper discontinuity, it was crucial that $\mathrm{MCG}(\mathcal{S}_g)$ acts on $\mathrm{Teich}(\mathcal{S}_g)$ by isometries. We will write

$$\mathrm{d}_{\mathcal{M}}(X,Y) = \min_{[f] \in \mathrm{MCG}(\mathcal{S}_g)} \mathrm{d}_{\mathrm{Teich}}([f] \cdot [X],[Y])$$

for the induced metric on $\mathcal{M}(\mathcal{S}_g)$. This makes Moduli space a metric space. The next chapter is concerned with compactness of subsets of $\mathcal{M}(\mathcal{S}_g)$.

[3] Meaning that the complement of any two representatives (one in each isotopy class) is a collection of disks.

1.3 The Thick Part and the End of Moduli Space

The first goal of this chapter is to state and prove *Mumford's compactness criterion.* Talking about a compactness criterion would be useless if the space in consideration was compact. So let us first verify that $\mathcal{M}(\mathcal{S}_g)$ is not compact. Given a hyperbolic surface $X \in \mathcal{M}(\mathcal{S}_g)$, let $L_{\min}(X)$ denote the infimum over all lengths of essential geodesics in X. A geodesic realizing $L_{\min}(X)$ must necessarily be simple. By lemma 1.8, $L_{\min}(X)$ is actually a minimum, is strictly positive, and is achieved by some geodesic.

Proposition 1.13. *$\mathcal{M}(\mathcal{S}_g)$ is not compact.*

Proof. Given a representative $[(X, \phi)] \in \mathrm{Teich}(\mathcal{S}_g)$ of $X \in \mathcal{M}(\mathcal{S}_g)$, take a geodesic γ in $\mathcal{S}_g$ such that $\phi(\gamma)$ realizes $L_{\min}(X)$, i.e. $L_{[X]}(\gamma) = L_{\min}(X)$. Fix a pants decomposition of $\mathcal{S}_g$ consisting of γ and $3g-4$ other geodesics, which are necessarily at least as long as γ. Denote by $(L_{\min}(X), l_2, \dots, \theta_{3g-3})$ the FN coordinate of $[(X, \phi)]$. As the Fenchel-Nielsen coordinates are a bijection, $(L_{\min}(X)/t, l_2, \dots, \theta_{3g-3})$ gives us another element of the Teichmüller space, for any $t > 1$. Denote the new element by $[X_t]$, and note that $L_{[X_t]}(\gamma) = L_{\min}(X)/t$, by construction. Next, consider the element $X_t \in \mathcal{M}(\mathcal{S}_g)$ obtained by projecting $[X_t]$ onto $\mathcal{M}(\mathcal{S}_g)$. We claim that $\mathrm{d}_{\mathcal{M}}(X, X_t) \to \infty$ as $t \to \infty$. If this was not true, then there exists a $K > 1$ such that, for all $t > 1$, there is some $[f_t] \in \mathrm{MCG}(\mathcal{S}_g)$ with

$$\mathrm{d}_{\mathcal{M}}(X, X_t) = \mathrm{d}_{\mathrm{Teich}}([f_t] \cdot [X], [X_t]) < \log(K)/2.$$

Corollary 1.11 tells us that

$$\frac{L_{[f_t]\cdot[X]}([\gamma])}{K} \leq L_{[X_t]}([\gamma]) \leq K \cdot L_{[f_t]\cdot[X]}([\gamma]).$$

Thus, by remark 1.1 and the definition of $L_{\min}(X)$,

$$\frac{L_{\min}(X)}{K} \leq \frac{L_{[X]}([f_t^{-1}(\gamma)])}{K} = \frac{L_{[f_t]\cdot[X]}([\gamma])}{K} \leq L_{[X_t]}([\gamma]) = \frac{L_{\min}(X)}{t},$$

for any $f_t \in [f_t]$, which is not true for large t. This proves that $\mathcal{M}(\mathcal{S}_g)$ is not compact. □

Along the way, we already showed *how* a sequence can leave any compact set in $\mathcal{M}(\mathcal{S}_g)$, namely, by pinching (at least one) essential geodesic. Mumford's compactness criterion verifies that this is the only way to leave any compact set. More precisely, it states the following:

Theorem 1.14 (Mumford)**.** *For any $\epsilon > 0$, the space*

$$\mathcal{M}_\epsilon(\mathcal{S}_g) = \{X \in \mathcal{M}(\mathcal{S}_g) \mid L_{\min}(X) \geq \epsilon\}$$

is compact.

We call $\mathcal{M}_\epsilon(\mathcal{S}_g)$ the *epsilon thick part of* $\mathcal{M}(\mathcal{S}_g)$. Note that $\mathcal{M}_\epsilon(\mathcal{S}_g)$, $\epsilon > 0$, is an *exhaustion* of $\mathcal{M}(\mathcal{S}_g)$, meaning that

$$\mathcal{M}(\mathcal{S}_g) = \bigcup_{\epsilon > 0} \mathcal{M}_\epsilon(\mathcal{S}_g).$$

Indeed, lemma 1.8 asserts that $\mathcal{M}(\mathcal{S}_g)$ is a subset of the union on the right hand side. In the proof of Mumford's compactness criterion, we will use the next theorem due to Bers. Given a hyperbolic surface X, let $B(X)$ be the minimal number such that X admits a pants decomposition $\{\gamma_1, \dots, \gamma_{3g-3}\}$ with $L_X(\gamma_n) \leq B(X)$, for all $1 \leq n \leq 3g-3$. We define *Bers' constant* of $\mathcal{S}_g$ as

$$\mathrm{B}_g = \sup_{X \in \mathcal{M}(\mathcal{S}_g)} B(X).$$

Theorem 1.15 (Bers)**.** *Bers' constant* B_g *is finite.*

We follow [5, p. 125ff.].

Proof. Note that any $3g-3$ essential disjoint geodesics in $X \in \mathcal{M}(\mathcal{S}_g)$ give rise to a pants decomposition of X. The statement will follow from the claim below, which we will prove by induction.
Claim: For any $1 \leq k \leq 3g-3$, there is a constant L_k depending only on g such that, for any $X \in \mathcal{M}(\mathcal{S}_g)$, there are k many essential disjoint geodesics in X of length at most L_k.
Let us establish the case $k=1$. Take $X \in \mathcal{M}(\mathcal{S}_g)$, and let γ denote the shortest geodesic there is in X. Then the injectivity radius is exactly $L_X(\gamma)/2$ (see proposition A.3). Thus, the disk $D(x,r)$ of radius r around any point $x \in X$ is isometrically embedded, for any $r < L_X(\gamma)/2$. We can compute the area of $D(x,r)$:

$$\operatorname{Area}(D(x,r)) = \int_0^{2\pi} \int_0^r \sinh(s) ds d\theta = 2\pi(\cosh(r)-1).$$

Together with the observation from the Gauss-Bonnet formula, we see that

$$2\pi(\cosh(r)-1) = \operatorname{Area}(D(x,r)) \leq \operatorname{Area}(X) = 4\pi(g-1).$$

We can let r limit to $L_X(\gamma)/2$ to get a logarithmic bound on $L_X(\gamma)$ in terms of g. This finishes the case $k=1$. Now suppose the claim is true for some $1 \leq k < 3g-3$. Let $X \in \mathcal{M}(\mathcal{S}_g)$, and cut X along the geodesics from the induction hypothesis. As $k < 3g-3$, there is at least one component Y that is not a pair of pants. Denote the boundary components of Y by $\gamma_1, \ldots, \gamma_n$. Given $r > 0$, set

$$C(r) = \left\{ y \in Y \;\middle|\; \operatorname{distance}(y, \partial Y) \leq r \right\}.$$

By the collar lemma (theorem A.4), for small r, the set $C(r)$ is a disjoint union of half-collars with area forms $ds^2 + L_Y(\gamma_j)^2 \cosh^2(s) dt$, $1 \leq j \leq n$.[4] Thus, the area is

$$\operatorname{Area}(C(r)) = \sum_{j=1}^n \int_0^1 \int_0^r L_X(\gamma_j) \cosh(s) ds dt = \left(\sum_{j=1}^n L_X(\gamma_j) \right) \sinh(r).$$

We enlarge r as long as $C(r)$ still is a disjoint union of half collars. This cannot be done indefinitely as $\operatorname{Area}(C(r)) < \operatorname{Area}(X) < \infty$. Hence, we will reach some limit value r_*. The half collars are characterized by the property that any geodesic arcs perpendicular to γ_i are parallel. Indeed, if a neighborhood is isometric to $[0,r] \times \gamma_i$, no two geodesic arcs can intersect. Conversely, if no two geodesics arcs intersect, then we can enlarge the collar neighborhood $[0,r] \times \gamma_i$ as non-intersection is an open property. Therefore, for the limit value r_*, we can find two geodesic arcs α_1 and α_2, each perpendicular to some boundary component of Y, which meet each other. The concatenation $\alpha = \alpha_1 \circ \alpha_2^{-1}$ is a geodesic arc of length $2r_*$ perpendicular to ∂Y in its two endpoints. We distinguish between two cases, see figure 1.
Case 1: α meets ∂Y in two different components, which we call without loss of generality γ_1 and γ_2.
Case 2: α meets ∂Y in a single component, which we call without loss of generality γ_1.

[4] We apply the collar lemma to X, which has no boundary, and then restrict to Y.

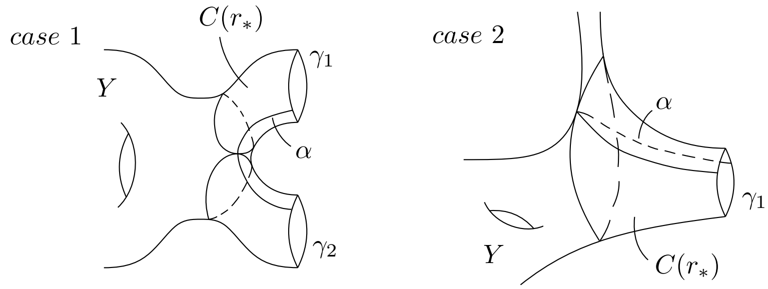

Figure 1: Collars meeting in a limit point.

Let us consider the first case. We denote by γ_{k+1} the geodesic representative of the concatenation $\alpha \circ \gamma_2^{-1} \circ \alpha^{-1} \circ \gamma_1$. By theorem A.2, γ_{k+1} does not intersect ∂Y. Denote the number $\sum_{j=1}^{n} L_X(\gamma_j)$ by L_X. Since the area form of the half-collars is $ds^2 + L_Y(\gamma_j)^2 \cosh^2(s)dt$, $1 \leq j \leq n$, the length of $\partial C(r) \setminus \partial Y$ is exactly $L_X \cosh(r)$. To shorten notation, define $l(r)$ to be the length of $\partial C(r) \setminus \partial Y$. If L_k is the bound from the induction hypothesis, then let M be the number $2L_k$, which depends only on g. If $r_* \leq \operatorname{arccosh}(2)$, then

$$L_X(\gamma_{k+1}) \leq l(r_*) = L_X \cosh(r_*) \leq M,$$

since γ_{k+1} is isotopic to the concatenation of the two curves in $\partial C(r_*) \setminus \partial Y$ that cause the limit case, and we are done. Now suppose $r_* > \operatorname{arccosh}(2)$. Then there is some $0 < r' < r_*$ with $r' = \operatorname{arccosh}(2)$. We compute

$$\begin{aligned} 4\pi(g-1) &\geq \operatorname{Area}(C(r_*)) - \operatorname{Area}(C(r')) = \int_0^1 \int_{r'}^{r_*} L_X \cosh(s) ds dt = L_X(\sinh(r_*) - \sinh(r')) = \\ &= L_X\Big(\sinh(r_*)\cosh(r_* - r') + \cosh(r')\sinh(r_* - r') - \sinh(r')\Big) \geq \underbrace{L_X \cosh(r')}_{=M} \sinh(r_* - r'). \end{aligned}$$

Thus, we obtained the bound

$$r_* - r' \leq \operatorname{arcsinh}\left(\frac{4\pi(g-1)}{M}\right),$$

which depends only on g. Since γ_{k+1} is isotopic to $\alpha \circ \gamma_2^{-1} \circ \alpha^{-1} \circ \gamma_1$, it is also isotopic to the curve that runs through the two curves in $\partial C(r') \setminus \partial Y$, parallel to the ones that cause the limit case, joined by the straight line from $\partial C(r')$ to $\partial C(r_*)$, see the red line in figure 2.

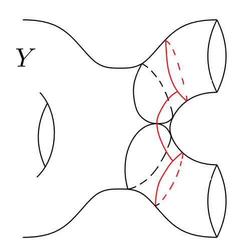

Figure 2: Isotopic curve in the collars.

This red line has length bounded by $L_X \cosh(r') + 4(r_* - r') \leq M + 4(r_* - r')$, which depends only on g. This finishes the first case. In the second case, α has two endpoints on γ_1 and splits it into two geodesic arcs, which we call β_1 and β_2. Then the concatenations $\alpha \circ \beta_1$ and $\alpha \circ \beta_2^{-1}$ are two closed curves. Let δ_1 and δ_2 be the geodesic representatives of these curves, see figure 3. Then these two together with the boundary component γ_1 bound a pair of pants in Y. If both δ_1 and δ_2 were boundary components of Y, then Y would already be a pair of pants, which we excluded. Thus, at least one of them is not a boundary component, and we define this geodesic to be γ_{k+1} (in the picture, it would be δ_2). A bound for the length of γ_{k+1} that depends only on g is obtained just as in the first case. This finishes the second case and, hence, the proof.

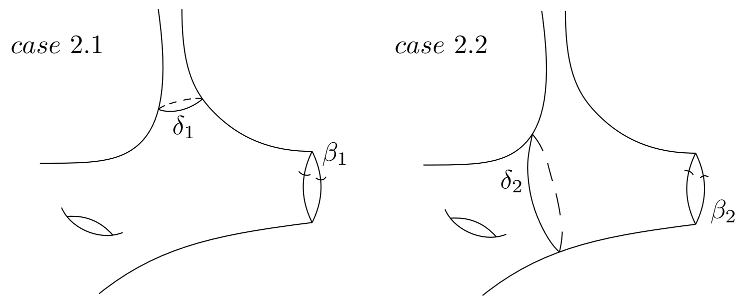

Figure 3: Isotopy classes in the second case.

□

Remark 1.16. *In [5, p. 125ff.], the length estimates are made more carefully, and the author is able to get a bound of linear growth order on Bers' constant.*

We can now prove Mumford's compactness criterion. Proving Bers' theorem was the main step towards this.

Proof of theorem 1.14. Let $\{X_n\}_{n\geq 1}$ be a sequence in $\mathcal{M}_\epsilon(\mathcal{S}_g)$, and let $[(X_n, \phi_n)] \in \mathrm{Teich}(\mathcal{S}_g)$ be some lift of X_n. Bers' theorem gives us pants decompositions $\mathcal{P}'_n$ of X_n with L_{X_n}-lengths at most B_g (and at least ϵ). Any $\mathcal{P}'_n$ can be pulled back to a pants decomposition $\mathcal{P}_n = \phi_n^{-1}(\mathcal{P}'_n) = \{\gamma_1^n, \ldots, \gamma_{3g-3}^n\}$ of $\mathcal{S}_g$ of $L_{[X_n]}$-length at most B_g (and at least ϵ). Since there are, up to the action of $\mathrm{MCG}(\mathcal{S}_g)$, only finitely pair of pants decompositions of $\mathcal{S}_g$, we can pass to a subsequence of $\{X_n\}_{n\geq 1}$ and take a sequence $\{[f_n]\}_{n\geq 1} \subset \mathrm{MCG}(\mathcal{S}_g)$ such that $[f_n](\mathcal{P}_n) = \mathcal{P}_1$ is the same pants decomposition for all $n \geq 1$. We can fix $\mathcal{P}_1$ as a frame for Fenchel-Nielsen coordinates. Since $[f_n](\mathcal{P}_n) = \mathcal{P}_1$, the FN coordinates of every $[f_n] \cdot [(X_n, \phi_n)]$ are inside $[\epsilon, L] \times \mathbb{R}^{3g-3}$. Indeed, for all $1 \leq i \leq 3g-3$, we compute

$$L_{[f_n]\cdot[X_n]}(\gamma_i^1) = L_{[X_n]}([f_n^{-1}(\gamma_i^1)]) = L_{[X_n]}(\gamma_i^n) \in [\epsilon, L],$$

where $f_n \in [f_n]$. For each n, let T_n denote the composition of Dehn twists, that sends $[f_n]\cdot[(X_n, \phi_n)]$ into $[\epsilon, L] \times [0, 2\pi]^{3g-3}$. By compactness of $[\epsilon, L] \times [0, 2\pi]^{3g-3}$, the sequence $\{T_n \cdot ([f_n] \cdot [(X_n, \phi_n)])\}_{n\geq 1}$ has some convergent subsequence in $\mathrm{Teich}(\mathcal{S}_g)$. Note that Dehn twists are elements of $\mathrm{MCG}(\mathcal{S}_g)$. Therefore, $T_n \cdot ([f_n]\cdot[(X_n, \phi_n)])$ and $[(X_n, \phi_n)]$ are the same element after projection to $\mathcal{M}(\mathcal{S}_g)$. Since $\mathcal{M}(\mathcal{S}_g)$ inherits the metric from $\mathrm{Teich}(\mathcal{S}_g)$, the projected subsequence is a convergent subsequence of $\{X_n\}_{n\geq 1}$, which finishes the proof. □

As a first application, we can use Mumford's compactness criterion to prove the following result about the raw length spectrum sort of determining the underlying hyperbolic surface.

Proposition 1.17. *For any $X \in \mathcal{M}(\mathcal{S}_g)$, the set*

$$\{Y \in \mathcal{M}(\mathcal{S}_g) \mid \mathrm{RLS}(X) = \mathrm{RLS}(Y)\}$$

is finite.

Proof. Let $X \in \mathcal{M}(\mathcal{S}_g)$ with a lift $[X] \in \mathrm{Teich}(\mathcal{S}_g)$. First, consider a compact set $K \subset \mathrm{Teich}(\mathcal{S}_g)$. We will show that the set

$$\{[Y] \in K \mid \mathrm{RLS}([X]) = \mathrm{RLS}([Y])\}$$

is finite. Since K is compact, it has finite diameter, denoted by D. Take $[Y] \in K$ with $\mathrm{RLS}([X]) = \mathrm{RLS}([Y])$. Then the dilatation of the Teichmüller map characterizing $\mathrm{d}_{\mathrm{Teich}}([X],[Y])$ is bounded by e^{2D}. Theorem 1.2 yields $9g-9$ geodesics $\gamma_1,\dots,\gamma_{9g-9}$ in $\mathcal{S}_g$ whose lengths uniquely characterize a point in $\mathrm{Teich}(\mathcal{S}_g)$. To shorten notation, set $L = \max_{1\le k\le 9g-9} L_{[X]}([\gamma_k])$. The corollary 1.11 to Wolpert's lemma implies the bound

$$L_{[Y]}([\gamma_k]) \le e^{2D}L, \text{ for all } 1 \le k \le 9g-9.$$

Lemma 1.8 asserts that

$$\mathrm{RLS}([Y]) \cap [0, e^{2D}L] = \mathrm{RLS}([X]) \cap [0, e^{2D}L]$$

is finite. Hence, there are only finitely many possibilities for $L_{[Y]}([\gamma_k])$, $1 \le k \le 9g-9$, which shows that

$$\{[Y] \in K \mid \mathrm{RLS}([X]) = \mathrm{RLS}([Y])\}$$

is finite. We now want to show that

$$\{Y \in \mathcal{M}(\mathcal{S}_g) \mid \mathrm{RLS}(X) = \mathrm{RLS}(Y)\}$$

is finite, as well. We already noted that

$$\mathcal{M}(\mathcal{S}_g) = \bigcup_{\epsilon>0} \mathcal{M}_\epsilon(\mathcal{S}_g)$$

as a consequence of lemma 1.8. Thus, $X \in \mathcal{M}_\epsilon(\mathcal{S}_g)$ for some $\epsilon > 0$. If $\mathrm{RLS}(X) = \mathrm{RLS}(Y)$ for $Y \in \mathcal{M}(\mathcal{S}_g)$, then we also have $Y \in \mathcal{M}_\epsilon(\mathcal{S}_g)$. The proof of Mumford's compactness criterion revealed that any $Y \in \mathcal{M}_\epsilon(\mathcal{S}_g)$ has a lift in $[\epsilon, \mathrm{B}_g]^{3g-3} \times [0,2\pi]^{3g-3}$. Hence, we can use the first step for $K = [\epsilon, \mathrm{B}_g]^{3g-3} \times [0,2\pi]^{3g-3}$ and conclude the theorem. □

In the remainder of this chapter, we want to study the topology at infinity of $\mathcal{M}(\mathcal{S}_g)$, meaning that we try to get some insights on the complement of any thick part $\mathcal{M}(\mathcal{S}_g) \setminus \mathcal{M}_\epsilon(\mathcal{S}_g)$. This proposition is our main tool to do so.

Proposition 1.18. *Suppose $[X],[Y] \in \mathrm{Teich}(\mathcal{S}_g)$ get projected to $\mathcal{M}(\mathcal{S}_g) \setminus \mathcal{M}_\epsilon(\mathcal{S}_g)$. Then we can find a path from $[X]$ to $[Y]$ that also gets projected to $\mathcal{M}(\mathcal{S}_g) \setminus \mathcal{M}_\epsilon(\mathcal{S}_g)$.*

Proof. By hypothesis, we can find two essential geodesics α and β in $\mathcal{S}_g$ with $L_{[X]}(\alpha) < \epsilon$ and $L_{[Y]}(\beta) < \epsilon$. By theorem A.6, there are finitely many geodesics $\alpha = \gamma_1, \gamma_2, \dots, \gamma_{n-1}, \gamma_n = \beta$ such that γ_i and γ_{i+1} do not intersect, for all $1 \le i \le n-1$. We claim that, for every $1 \le i \le n$, there exists an element $[X_i] \in \mathrm{Teich}(\mathcal{S}_g)$ with $L_{[X_i]}(\gamma_i) < \epsilon$ and such that there exists a path from $[X]$ to $[X_i]$ that gets projected to $\mathcal{M}(\mathcal{S}_g) \setminus \mathcal{M}_\epsilon(\mathcal{S}_g)$. For $i = 1$ this is clear, since $\gamma_1 = \alpha$. Given such $[X_i]$, take a pants decomposition containing γ_i and γ_{i+1} for a Fenchel-Nielsen frame with γ_i and γ_{i+1} in the first two coordinates. In this frame, $[X_i]$ takes the form $(l_1, \dots, \theta_{3g-3})$ with $l_1 = L_{[X_i]}(\gamma_i) < \epsilon$ and $l_2 = L_{[X_i]}(\gamma_{i+1})$. Consider the element $[X_t] \in \mathrm{Teich}(\mathcal{S}_g)$ given by the FN-coordinate $(l_1, \frac{l_2}{t}, l_3, \dots, \theta_{3g-3})$. Then $L_{\min}(X_t) \leq l_1 < \epsilon$ so that $[X_t]$ projects to $\mathcal{M}(\mathcal{S}_g) \setminus \mathcal{M}_\epsilon(\mathcal{S}_g)$. Take T so large that $l_2/T < \epsilon$, and define $[X_{i+1}] = [X_T]$. Since $L_{[X_{i+1}]}(\gamma_{i+1}) = l_2/T < \epsilon$, this finishes the proof of the claim. For the last step $i = n$, we can actually take $[X_n]$ to have $L_{[X_n]}(\gamma_n) = L_{[Y]}(\beta)$. Using a Fenchel-Nielsen frame containing β, we can construct a linear path between $[X_n]$ and $[Y]$. This path gets projected to $\mathcal{M}(\mathcal{S}_g) \setminus \mathcal{M}_\epsilon(\mathcal{S}_g)$ because the length coordinate in β will be constantly $L_{[Y]}(\beta) < \epsilon$. □

As a first immediate corollary we can specify the number of ends that $\mathcal{M}(\mathcal{S}_g)$ has.

Corollary 1.19. *$\mathcal{M}(\mathcal{S}_g)$ has exactly one end, which means that, for every compact set $K \subset \mathcal{M}(\mathcal{S}_g)$, the complement $\mathcal{M}(\mathcal{S}_g) \setminus K$ has exactly one connected component with non-compact closure.*

Proof. Let $K \subset \mathcal{M}(\mathcal{S}_g)$ be compact. Since

$$\mathcal{M}(\mathcal{S}_g) = \bigcup_{\epsilon > 0} \mathcal{M}_\epsilon(\mathcal{S}_g),$$

there is some $\epsilon > 0$ with $K \subset \mathcal{M}_\epsilon(\mathcal{S}_g)$. Indeed, otherwise there would be a sequence of elements $X_n \in K \setminus \mathcal{M}_{\frac{1}{n}}(\mathcal{S}_g)$ and, by the proof of Mumford's compactness criterion, each X_n would have a lift $[X_n]$ in $(0, \epsilon)^{3g-3} \times [0, 2\pi]^{3g-3}$ (for some fixed FN-coordinates). Since $\mathcal{M}(\mathcal{S}_g)$ inherits the quotient metric from $\mathrm{Teich}(\mathcal{S}_g)$, a converging subsequence of $(X_n)_{n \geq 0}$ in $\mathcal{M}(\mathcal{S}_g)$ would give rise to a converging subsequence of $([X_n])_{n \geq 0}$ in $\mathrm{Teich}(\mathcal{S}_g)$. However, this cannot happen. For the sets $\mathcal{M}_\epsilon(\mathcal{S}_g)$, we just showed that the complement is path-connected. □

Now let us briefly motivate why it is interesting to study the end of Moduli space. If we consider the entire space and ask ourselves what its fundamental group is, then the unsatisfying answer is that it is trivial.

Theorem 1.20. *$\mathcal{M}(\mathcal{S}_g)$ is simply connected.*

We prove this theorem further below. On the other hand, the *orbifold fundamental group* is not trivial. Given a path-connected topological space X, a group G acting properly discontinuous on X, and a base point $[x_0] \in X/G$, the orbifold fundamental group of the quotient X/G is defined as follows. Two paths in X/G are *homotopic in the orbifold sense* if there exist piecewise lifts that are homotopic to each other and such that the piecewise lifts can be attached together by the action of G, i.e. the endpoint of one piece and the starting point of the next piece lie on the same G-orbit. The orbifold fundamental group is the set of equivalence classes of loops based at $[x_0]$. We claim that a loop in X/G is trivial in the orbifold sense if and only if there is a lift that is a contractible loop in X. Indeed, if γ is trivial in the orbifold sense, then there are lifts $x_1, \dots, x_n$ of $[x_0]$ and piecewise lifts $\gamma_1, \dots, \gamma_n$ of γ such that each γ_j is a contractible loop based at x_j, $1 \leq j \leq n$. Since each γ_j is a loop, we actually could have taken lifts that all have the same base point $x_1 = \dots = x_n$. But then γ has a global (not just piecewise) lift that is a contractible loop based at x_1. The other direction is obvious. Now consider the case where X is simply connected. Then the piecewise lifts are automatically homotopic if their start- and endpoints agree. Hence, in this case, two paths in X/G are homotopic in the orbifold sense if there exist lifts of the paths with the same start- and endpoint. Therefore, if X is simply connected, then the orbifold fundamental group of X/G simply is G. Thus, the orbifold fundamental group of $\mathcal{M}(\mathcal{S}_g) = \mathrm{Teich}(\mathcal{S}_g)/\mathrm{MCG}(\mathcal{S}_g)$ is the Mapping Class Group. The next corollary tells us that the interesting bits of the orbifold fundamental group take place at the end of Moduli space.

Corollary 1.21. *For any $\epsilon > 0$, the inclusion map $\mathcal{M}(\mathcal{S}_g) \setminus \mathcal{M}_\epsilon(\mathcal{S}_g) \hookrightarrow \mathcal{M}(\mathcal{S}_g)$ induces a surjection of orbifold fundamental groups.*

Proof. Take a point in $\mathcal{M}(\mathcal{S}_g) \setminus \mathcal{M}_\epsilon(\mathcal{S}_g)$ as a base point. Let γ be a loop in $\mathcal{M}(\mathcal{S}_g)$ based at that point, and lift it to a path γ' in $\mathrm{Teich}(\mathcal{S}_g)$. Proposition 1.18 provides us with a path δ' in $\mathrm{Teich}(\mathcal{S}_g)$ connecting the two endpoints of γ' that gets projected to a path δ in $\mathcal{M}(\mathcal{S}_g) \setminus \mathcal{M}_\epsilon(\mathcal{S}_g)$. Then δ and γ are homotopic in the orbifold sense in $\mathcal{M}(\mathcal{S}_g) \setminus \mathcal{M}_\epsilon(\mathcal{S}_g)$, by definition in the case of a simply connected space. □

We end this section by providing a proof that Moduli space is simply connected. The approach we use is based on [10, p. 86]. The theorem will be deduced from a more general result about fundamental groups of orbifolds, which is taken from [2, p. 299].

Theorem 1.22. *Let X be a path-connected, simply connected, locally compact, metric space. Suppose G is a subgroup of the group of homeomorphisms of X to itself such that its action on X is properly discontinuous. Denote by H the smallest normal subgroup of G containing every element in G that has a fixed point. Then the fundamental group of the orbifold X/G is the factor group G/H.*

If we take X to be Teichmüller space and G to be the Mapping Class Group, then simple connectivity of Moduli space is an immediate consequence of the above theorem together with the lemma below. Indeed, the theorem asserts that the fundamental group of $\mathcal{M}(\mathcal{S}_g)$ is $\mathrm{MCG}(\mathcal{S}_g)/H$, where H is the smallest normal subgroup of $\mathrm{MCG}(\mathcal{S}_g)$ that contains every element in $\mathrm{MCG}(\mathcal{S}_g)$ that has a fixed point. Furthermore, it is proved in [3] that $\mathrm{MCG}(\mathcal{S}_g)$ is generated by three elements of finite order. Thus, if we show that every mapping class of finite order has a fixed point in $\mathrm{Teich}(\mathcal{S}_g)$, then the subgroup H is the entire Mapping Class Group, proving that $\mathcal{M}(\mathcal{S}_g)$ has trivial fundamental group.

Lemma 1.23. *Every element in* $\mathrm{MCG}(\mathcal{S}_g)$ *of finite order has a fixed point in* $\mathrm{Teich}(\mathcal{S}_g)$.

Proof. Suppose $[f] \in \mathrm{MCG}(\mathcal{S}_g)$ has order $n < \infty$. Consider the cyclic subgroup $\langle f \rangle$ of $\mathrm{MCG}(\mathcal{S}_g)$. If no element of $\langle f \rangle$ had a fixed point in $\mathrm{Teich}(\mathcal{S}_g)$, then theorem 1.22 would imply that the quotient $\mathrm{Teich}(\mathcal{S}_g)/\langle f \rangle$ had fundamental group $\langle f \rangle \simeq \mathbb{Z}/n\mathbb{Z}$ and, hence, would be an Eilenberg-MacLane $K(\mathbb{Z}/n\mathbb{Z}, 1)$-space. This is a contradiction because the homology groups of $\mathrm{Teich}(\mathcal{S}_g)/\langle f \rangle$ necessarily vanish in degree greater than $6g-6$, but all $K(\mathbb{Z}/n\mathbb{Z}, 1)$-spaces are homotopy equivalent (see [8, p. 366]) and there are other such spaces with infinitely many non-trivial homology groups (for instance, the infinite lens space $S^\infty/(\mathbb{Z}/n\mathbb{Z})$, see [8, p. 304]). Thus, we have $[f^k] \cdot [X] = [X]$, for some $1 \le k < n$ and some $[X] \in \mathrm{Teich}(\mathcal{S}_g)$. If n is prime, then $[f]$ is a power of $[f^k]$ and, therefore, fixes $[X]$, which we wanted to show. Now assume n has prime factorization $n = p_1 \cdots p_s$. We proceed by induction on s. The case $s = 1$ is already dealt with so that we may assume $s \ge 2$. The mapping class $[\tilde{f}] = [f^{p_s}]$ has order $p_1 \cdots p_{s-1}$, and, by induction hypothesis, it fixes some point $[X] \in \mathrm{Teich}(\mathcal{S}_g)$. Let $\mathrm{Fix}([\tilde{f}]) \subset \mathrm{Teich}(\mathcal{S}_g)$ denote the set of fixed points of $[\tilde{f}]$. Since $[f]$ and $[\tilde{f}]$ commute, $\langle f \rangle$ acts on $\mathrm{Fix}([\tilde{f}])$. The action of $\langle \tilde{f} \rangle < \langle f \rangle$ on $\mathrm{Fix}([\tilde{f}])$ clearly lies in the kernel. Thus, we get an action on $\mathrm{Fix}([\tilde{f}])$ by the quotient $\langle f \rangle / \langle \tilde{f} \rangle$. The latter is isomorphic to $\mathbb{Z}/p_s\mathbb{Z}$. We claim that $\mathrm{Fix}([\tilde{f}])$ is homeomorphic to the Teichmüller space of some surface $\mathcal{S}$. Then we can use the argument for n prime on $\langle f \rangle / \langle \tilde{f} \rangle$ acting on $\mathrm{Fix}([\tilde{f}]) \simeq \mathrm{Teich}(\mathcal{S})$ to conclude that some element in $\langle f \rangle / \langle \tilde{f} \rangle$ other than the identity fixes a point in $\mathrm{Teich}(\mathcal{S}) \simeq \mathrm{Fix}([\tilde{f}]) \subset \mathrm{Teich}(\mathcal{S}_g)$.[5] Since $\langle f \rangle / \langle \tilde{f} \rangle$ is just the group $\langle f^{n/p_s} \rangle$, we have found an element of the form $[f^{ln/p_s}]$, $1 \le l < p_s$, fixing some $[Y] \in \mathrm{Fix}([\tilde{f}])$. Since p_s is prime, some power of $[f^{ln/p_s}]$ equals $[f^{n/p_s}]$. Then $[f^{n/p_s}]$ fixes $[Y]$, which is also fixed by $[\tilde{f}] = [f^{p_s}]$. Hence, we have found a fixed point of $[f^n] = [f]$. It only remains to show the claim that $\mathrm{Fix}([\tilde{f}]) \simeq \mathrm{Teich}(\mathcal{S})$ for some surface $\mathcal{S}$. However, we postpone proving this to lemma 1.24 in the next subchapter because we first need to discuss non-compact surfaces. □

Let us quickly observe the converse that any mapping class with a fixed point must have finite order. Indeed, suppose $[f]$ fixes $[(X, \phi)]$. Then $\phi \circ f \circ \phi^{-1}$ is isotopic to an isometry I of X. Since $\mathrm{Isom}^+(X)$ is finite, I has finite order, say n. Hence, f^n is isotopic to $\phi^{-1} \circ I^n \circ \phi = \mathrm{id}_{\mathcal{S}_g}$, and the order of f must be a divisor of n and, in particular, must be finite.

[5] The surface $\mathcal{S}$ will not be compact, but the argument goes through; we will discuss the Teichmüller space of a non-compact surface in the next subchapter.

1.4 Compactification of Moduli Space

In this section, we want to compactify $\mathcal{M}(\mathcal{S}_g)$. Mumford's compactness criterion already gives us a hint on how to do that. Namely, we need to extend $\mathcal{M}(\mathcal{S}_g)$ by surfaces that have an "essential geodesic of length zero". To do this, we first work in $\mathrm{Teich}(\mathcal{S}_g)$ and introduce the *augmented Teichmüller space* $\overline{\mathrm{Teich}(\mathcal{S}_g)}$. The surfaces we want to introduce will have node points as depicted in figure 4 below. To begin with, we will briefly discuss Teichmüller theory for non-compact surfaces. These are obtained by taking a surface $\mathcal{S}_g$ (or $\mathcal{S}_{g,b}$, but we are not interested in this case) and removing finitely many, say n, distinct points. The new surface will be denoted by $\mathcal{S}_{g,n}$. The Euler characteristic of $\mathcal{S}_{g,n}$ is $2-2g-n$ and, hence, for $g \geq 1$ and $(g,n) \neq (1,0)$, $\mathcal{S}_{g,n}$ still admits complete hyperbolic metrics. Therefore, the second view on $\mathrm{Teich}(\mathcal{S}_g)$ via hyperbolic metrics is also valid for $\mathrm{Teich}(\mathcal{S}_{g,n})$. We can adapt the first view on $\mathrm{Teich}(\mathcal{S}_g)$, as well, with only adding the remark that the homotopy from the composition of two markings to an isometry is allowed to permute the removed points. In order to give $\mathrm{Teich}(\mathcal{S}_{g,n})$ a topology, we use the following trick. Given $\mathcal{S}_{g,n}$, we consider the surface $\mathcal{S}_{g,b}$, $b = n$, where we removed small disks around the n removed points. For this surface, we have a homeomorphism $\mathrm{Teich}(\mathcal{S}_{g,b}) \to \mathbb{R}^{3g-3+2b}_{>0} \times \mathbb{R}^{3g-3+b}$. Now we view $\mathcal{S}_{g,n}$ as the surface obtained by letting the lengths of the boundaries tend to zero. This gives us a bijection $\mathrm{Teich}(\mathcal{S}_{g,n}) \to \mathbb{R}^{3g-3+n}_{>0} \times \mathbb{R}^{3g-3+n}$, which we can define to be a homeomorphism. Before proceeding with defining noded surfaces, let us fill in the remainder of the proof of lemma 1.23.

Lemma 1.24. *If* $[\tilde{f}] \in \mathrm{MCG}(\mathcal{S}_g)$ *has a non-empty set of fixed points* $\mathrm{Fix}([\tilde{f}]) \subset \mathrm{Teich}(\mathcal{S}_g)$*, then it is homeomorphic to* $\mathrm{Teich}(\mathcal{S})$ *for some surface* $\mathcal{S}$*.*

Proof. Let us assume that $[\tilde{f}]$ is not the identity element. Suppose $[\tilde{f}]$ fixes $[(X,\phi)]$. This means that, for any representative $\tilde{f} \in [\tilde{f}]$, $\phi \circ \tilde{f} \circ \phi^{-1}$ is isotopic to a unique isometry $\Phi\colon X \to X$, which does not depend on the choice of representative. Let us switch to the complex setting and regard Φ as a conformal map of the Riemann surface X. We can consider the quotient of X by the action of Φ. The quotient space will have a well-defined complex structure everywhere except at the (finitely many) fixed points of Φ. Thus, by removing these fixed points, we obtain a complex surface of topological type $\mathcal{S}_{g,n}$, which we denote by $\mathcal{S}$. We construct a map from $\mathrm{Fix}([\tilde{f}])$ to $\mathrm{Teich}(\mathcal{S})$ as follows, where $\mathrm{Teich}(\mathcal{S})$ is understood in the complex setting discussed in the first subchapter. Let $[(Y,\psi)] \in \mathrm{Fix}([\tilde{f}])$ be given. As before, $\psi \circ \tilde{f} \circ \psi^{-1}$ is isotopic to a unique conformal map $\Psi\colon Y \to Y$. Let $h\colon X \to Y$ be the Teichmüller map from theorem 1.4 that is isotopic to the change-of-marking map $c = \psi \circ \phi^{-1}$. Note that

$$\Psi \simeq c \circ \phi \circ \tilde{f} \circ \phi^{-1} \circ c^{-1} \simeq c \circ \Phi \circ c^{-1}$$

so that $\Psi^{-1} \circ h \circ \Phi$ is isotopic to c. Since Φ and Ψ are conformal maps, $\Psi^{-1} \circ h \circ \Phi$ has the same dilatation as h. By the uniqueness statement in Teichmüller's theorem, the latter two maps are equal, i.e. $h \circ \Phi = \Psi \circ h$. Hence, if $\mathcal{S}_Y$ denotes the complex surface obtained from taking the quotient of Y by the action of Ψ and removing the fixed points, then h factors to a map $h_Y : \mathcal{S} \to \mathcal{S}_Y$. In particular, $[(\mathcal{S}_Y, h_Y)]$ defines an element in $\mathrm{Teich}(\mathcal{S})$. We just constructed a well-defined map $\mathrm{Fix}([\tilde{f}]) \to \mathrm{Teich}(\mathcal{S})$. Injectivity of this map is a consequence of uniqueness of the Teichmüller map. We need to check that the map is surjective, as well. Assume that an arbitrary point $[(\tilde{\mathcal{S}}, \tilde{j})] \in \mathrm{Teich}(\mathcal{S})$ is given. Take a Riemann surface Y homeomorphic to $\mathcal{S}_g$ and a conformal map $\Psi\colon Y \to Y$ such that the quotient of Y by the action of Ψ minus the fixed points is exactly the complex manifold $\tilde{\mathcal{S}}$. Then $\tilde{j}$ is a map from the quotient of X by Φ to the quotient of Y by Ψ, and we can lift it to a map $j\colon X \to Y$ with $\Psi \circ j = j \circ \Phi$. Define the map ψ to be the composition $j \circ \phi\colon \mathcal{S}_g \to Y$. This way, $[(Y,\psi)]$ defines an element in $\mathrm{Teich}(\mathcal{S}_g)$. Moreover, we see that

$$\psi^{-1} \circ \Psi \circ \psi = \phi^{-1} \circ j^{-1} \circ \underbrace{\Psi \circ j}_{=j\circ\Phi} \circ\, \phi = \phi^{-1} \circ \Phi \circ \phi \in [\tilde{f}],$$

which shows that $[(Y,\psi)]$ is an element in $\mathrm{Fix}([\tilde{f}])$. Furthermore, the change-of-marking map $\psi \circ \phi^{-1}$ is exactly j so that the Teichmüller map $h\colon X \to Y$ that is isotopic to the change-of-marking map is actually isotopic to j. Thus, the factor map of h is isotopic to $\tilde{j}$, which shows that $[(\tilde{\mathcal{S}},\tilde{j})]$ and $[(\mathcal{S}_Y,h_Y)]$ are the same element in $\mathrm{Teich}(\mathcal{S})$. This concludes surjectivity. Continuity of the map as well as continuity of the inverse follow immediately from corollary 1.5 because, in the construction, we only used Teichmüller maps isotopic to change-of-marking maps and their quotient maps. $\square$

Let us return to the goal of compactifying Moduli space and introduce the notion of a node point. Switching back to the setting of Riemann surfaces, we characterize a *node point* by saying that it has a neighborhood that is biholomorphic to the set $\{(z,w)\in\mathbb{C}^2 \mid |z|<1, |w|<1, zw=0\}$, where the biholomorphic map sends the node point to $(0,0)$.

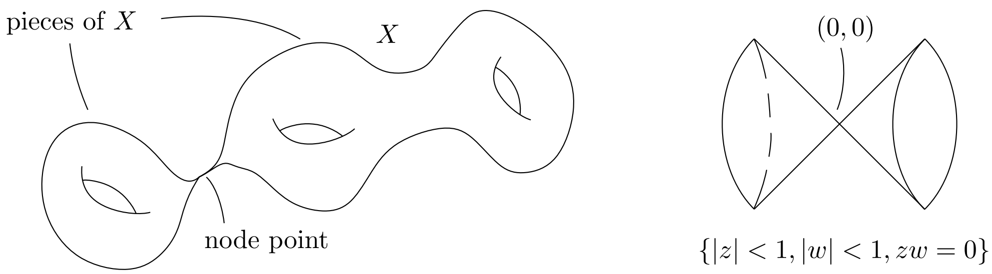

Figure 4: A hyperbolic surface with a node point.

We call X a *hyperbolic surface with nodes* if X minus its node points is homeomorphic to a collection of disjoint Riemann surfaces of type $\mathcal{S}_{g,n}$ with $\chi(\mathcal{S}_{g,n})<0$. The components of X minus its node points are called its *pieces*. In the construction of $\mathrm{Teich}(\mathcal{S}_g)$, we introduced marked hyperbolic surfaces. The analogue of a marking for surfaces with nodes is the following. Fix a pants decomposition $(\gamma_1,\dots,\gamma_{3g-3})$ of $\mathcal{S}_g$. A *marking* of a hyperbolic surface X *with nodes at* $\gamma_{i_1},\dots,\gamma_{i_m}$ is a continuous map $\phi\colon \mathcal{S}_g \to X$ such that the restriction of ϕ to $\mathcal{S}_g \setminus \{\gamma_{i_1},\dots,\gamma_{i_m}\}$ is a homeomorphism onto the pieces of X. Analogous to before, we say two marked hyperbolic surfaces with nodes (X_1,ϕ_1) and (X_2,ϕ_2) are *homotopic* if there exists a continuous map $I\colon X_1\to X_2$ that is an isometry when restricted to any piece of X_1 such that $I\circ\phi_1$ and ϕ_2 are isotopic.[6]

Remark 1.25. *Note that requiring the restrictions to each piece X_k to be homotopic markings in* $\mathrm{Teich}(\mathcal{S}_{g,n_k})$ *is not equivalent to the given definition of homotopic markings of hyperbolic surfaces with nodes, because the punctures of the piece X_k are allowed to get permuted, which would make the existence of a globally continuous map $I\colon X_1\to X_2$ impossible.*

Remark 1.26. *If X_k is a piece of X with n_k adjacent node points ("counted with multiplicity"), then $(X_k,\phi|_{X_k})$ is a marked hyperbolic structure on $\mathcal{S}_{g,n_k}$. Hence, we can associate to (X,ϕ) a tuple in*

$$\mathrm{Teich}(\mathcal{S}_{g,n_1})\times\cdots\times\mathrm{Teich}(\mathcal{S}_{g,n_l}),$$

where l denotes the number of pieces that X has.

We can define the *augmented Teichmüller space* $\overline{\mathrm{Teich}(\mathcal{S}_g)}$ by adding the equivalence classes of marked hyperbolic surfaces with nodes to $\mathrm{Teich}(\mathcal{S}_g)$. In Fenchel-Nielsen coordinates, we identify a marked hyperbolic surface with nodes at $(\gamma_{i_1},\dots,\gamma_{i_m})$ not with a single tuple in $\mathbb{R}^{3g-3}_{\geq 0}\times\mathbb{R}^{3g-3}$, but a subset. Namely, it will be the subset that has zero entries in the length coordinates of $\gamma_{i_1},\dots,\gamma_{i_m}$, has the entire

[6]This time around, we cannot simplify this to saying that $\phi_1\circ\phi_2^{-1}$ is isotopic to an isometry on each piece because ϕ_2 is not globally invertible.

axis $\mathbb{R}$ in the twist coordinates of $\gamma_{i_1}, \dots, \gamma_{i_m}$, and the obvious ones in the other coordinates. By the latter, we mean to consider the pants decomposition of the pieces induced by the pants decomposition $(\gamma_1, \dots, \gamma_{3g-3})$ of the unpinched surface and neglect the FN-entries for the punctures of each piece.

Remark 1.27. *Identifying the equivalence class of a noded surface with the mentioned subset is motivated as follows. The twist coordinates at the nodes are "arbitrary" since two surfaces with nodes that only differ in the twist coordinates at some nodes are homotopic in the sense of markings.*

We want to give $\overline{\text{Teich}(\mathcal{S}_g)}$ a topology that coincides with $\mathbb{R}^{3g-3}_{>0} \times \mathbb{R}^{3g-3}$ when restricted to $\text{Teich}(\mathcal{S}_g)$. Given a point $[X] \in \overline{\text{Teich}(\mathcal{S}_g)} \setminus \text{Teich}(\mathcal{S}_g)$ with nodes, we define its $(\epsilon, \delta) = (\epsilon_1, \dots, \epsilon_{3g-3}, \delta_1, \dots, \delta_{3g-3})$ neighborhood as the set of points $[Y] \in \overline{\text{Teich}(\mathcal{S}_g)}$ satisfying

$$\begin{aligned} |L_{[X]}(\gamma_k) - L_{[Y]}(\gamma_k)| &< \epsilon_k, \text{ for all } 1 \leq k \leq 3g-3, \\ |\theta_k([X]) - \theta_k([Y])| &< \delta_k, \text{ for all } k \text{ with } L_{[X]}(\gamma_k), L_{[Y]}(\gamma_k) \neq 0. \end{aligned}$$

Note that this does not give $\overline{\text{Teich}(\mathcal{S}_g)}$ the topology of $\mathbb{R}^{3g-3}_{\geq 0} \times \mathbb{R}^{3g-3}$. The last remark guarantees us that the new topology is Hausdorff. We can define the same action by $\text{Hom}^+(\mathcal{S}_g)$ on $\overline{\text{Teich}(\mathcal{S}_g)}$ as on $\text{Teich}(\mathcal{S}_g)$. It will again be invariant under $\text{Hom}_0(\mathcal{S}_g)$ and, hence, induce a well-defined action of $\text{MCG}(\mathcal{S}_g)$ on $\overline{\text{Teich}(\mathcal{S}_g)}$. However, we cannot speak of an action by isometries because the Teichmüller metric d_{Teich} is only defined on $\text{Teich}(\mathcal{S}_g)$. For a metric on $\overline{\text{Teich}(\mathcal{S}_g)}$, we will introduce the *Weil-Petersson metric* in the next chapter. As before, we set

$$\overline{\mathcal{M}(\mathcal{S}_g)} = \overline{\text{Teich}(\mathcal{S}_g)}/\text{MCG}(\mathcal{S}_g).$$

Since d_{Teich} is not defined on $\overline{\text{Teich}(\mathcal{S}_g)}$, the proof of Fricke's theorem (proper discontinuity of the action) does not extend to $\overline{\mathcal{M}(\mathcal{S}_g)}$. However, note that Mumford's compactness criterion still holds because the topology of $\overline{\mathcal{M}(\mathcal{S}_g)}$ restricted to $\mathcal{M}(\mathcal{S}_g)$ is the same as the topology of $\text{Teich}(\mathcal{S}_g)/\text{MCG}(\mathcal{S}_g)$. Lastly, let us formally state the goal of this chapter, which we just reached.

Theorem 1.28. $\overline{\mathcal{M}(\mathcal{S}_g)}$ *is compact.*

Proof. Take a sequence $\{X_n\}_{n\geq 1} \subset \overline{\mathcal{M}(\mathcal{S}_g)}$. After passing to a subsequence, we may distinguish between three cases. Firstly, we could have $\{X_n\}_{n\geq 1} \subset \mathcal{M}_\epsilon(\mathcal{S}_g)$ for some $\epsilon > 0$, secondly we could have $\{X_n\}_{n\geq 1} \subset \mathcal{M}(\mathcal{S}_g)$ without assuming that the sequence stays in some ϵ-thick part, and the third possibility is $\{X_n\}_{n\geq 1} \subset \overline{\mathcal{M}(\mathcal{S}_g)} \setminus \mathcal{M}(\mathcal{S}_g)$. In the first case, Mumford's compactness criterion gives us a convergent subsequence. In the second case, $L_{\min}(X_n)$ tends to zero. We want to find an element $Y \in \mathcal{M}(\mathcal{S}_g) \setminus \mathcal{M}_\epsilon(\mathcal{S}_g)$ and a subsequence of $\{X_n\}_{n\geq 1}$ that converges to X. The proof of Mumford's compactness criterion shows that, after passing to a subsequence, we can find a pants decomposition P of $\mathcal{S}_g$ for Fenchel-Nielsen coordinates and find lifts $[X_n]$ of X_n that lie in $(0, \text{B}_g] \times [0, 2\pi]$. Since the length of any geodesic in P is uniformly bounded from above by B_g, the width of their collars is uniformly bounded from below. The hypothesis that $L_{\min}(X_n)$ tends to zero asserts that, for all $\epsilon > 0$, there is some $n \geq 1$ and at least one geodesic in $\mathcal{S}_g$ that is shorter than ϵ in $L_{[X_n]}$-length. Since the width of the collars of geodesics in P is uniformly bounded from below, there is some $\epsilon_0 > 0$ such that, for all $\epsilon < \epsilon_0$ and all $n \geq 1$, we must have that all geodesics in $\mathcal{S}_g$ that are $L_{[X_n]}$-shorter than ϵ are elements of P. Let $Q \subset P$ contain all the geodesics γ_j in P for which the length coordinate $L_{[X_n]}(\gamma_j)$ is bounded away from zero uniformly in n. By passing to a subsequence, we may assume that each length and twist coordinate $L_{[X_n]}(\gamma_j)$, $\theta_j([X_n])$, $\gamma \in Q$, are converging to some values l_j and θ_j. Let $[Y] \in \overline{\text{Teich}(\mathcal{S}_g)} \setminus \text{Teich}(\mathcal{S}_g)$ denote the equivalence class of a hyperbolic surface of genus g with nodes at $P \setminus Q$ that has length and twist parameters l_j and θ_j in the other coordinates. By construction, $[X_n]$ converges to $[Y]$ and, hence, X_n converges to the projection of $[Y]$. This concludes the second case. Now suppose that $\{X_n\}_{n\geq 1} \subset \overline{\mathcal{M}(\mathcal{S}_g)} \setminus \mathcal{M}(\mathcal{S}_g)$. After passing to a subsequence, we can assume that every X_n has the same number of node points, say N. Moreover, we may also assume that every X_n has the same number of pieces. Let $X_n^1 \ldots, X_n^k$ denote the pieces of X_n. We argue in remark 1.29 below that the proof of Mumford's compactness criterion generalizes to the case of a punctured surface $\mathcal{S}_{g,n}$. Thus, we can apply the previous case to the sequences of pieces to obtain limit points $[Y^j]$ of each $\{X_n^j\}_{n\geq 1}$, $1 \leq j \leq k$. Let $[Y] \in \overline{\mathrm{Teich}(\mathcal{S}_g)}$ denote the equivalence class of a hyperbolic surface with N node points and pieces that are isometric to $[Y^j]$, $1 \leq j \leq k$. Pants decompositions of X_n^j glue together to define a pants decomposition of $\mathcal{S}_g$, where we insert geodesics where the nodes are. Using the glued pants decomposition for Fenchel-Nielsen coordinates on $\mathcal{S}_g$, $[X_n]$ converges to $[Y]$, by construction. This finishes the proof. $\square$

Remark 1.29. *Given a surface of type $\mathcal{S}_{g,n}$, we replace it with a surface of type $\mathcal{S}_{g,b}$, where we removed a very small disk around each puncture. Note that the proof of Bers' theorem goes through for surfaces of type $\mathcal{S}_{g,b}$ and, hence, so does the proof of Mumford's compactness criterion. Now we adapt the definition of $L_{\min}$ by allowing only non-peripheral geodesics. This way, the statement of Mumford's compactness criterion still holds if we collapse the boundary components to reattain the punctures.*

Remark 1.30. *$\overline{\mathcal{M}(\mathcal{S}_g)}$ is the Deligne-Mumford compactification of $\mathcal{M}(\mathcal{S}_g)$.*

This concludes the discussion of Moduli space. The next two chapters can be read (almost) independently of this chapter. We will now explore Teichmüller theory from the point of view of Riemann surfaces.

2 The Weil-Petersson Metric

2.1 Analytic Teichmüller Theory

Let $\mathbb{H}^*$ denote the lower half plane in $\mathbb{C}$. Throughout this chapter, Γ denotes a Fuchsian group, i.e. a discrete subgroup of $\mathrm{PSL}(2,\mathbb{R}) \simeq \mathrm{Aut}(\mathbb{H})$. The Fuchsian group Γ is taken such that the quotients $\mathbb{H}/\Gamma$ and $\mathbb{H}^*/\Gamma$ are closed Riemann surfaces. The *(Beltrami) coefficient of a quasi-conformal map f* is the function $\mu(f) = \overline{\partial} f/\partial f$. The following result is a direct consequence of the chain rule for the Wirtinger derivatives and will be used often throughout the chapter.

Lemma 2.1. *Whenever the composition is defined, for two quasi-conformal maps f and g, we have*

$$\mu(g \circ f^{-1}) \circ f = \frac{\partial f}{\overline{\partial f}} \frac{\mu(g) - \mu(f)}{1 - \mu(g)\overline{\mu(f)}}.$$

In particular,

$$\mu(g \circ f^{-1}) \circ f \equiv 0 \iff \mu(f) \equiv \mu(g).$$

Proof. With the chain rule, we can compute

$$\begin{aligned}
\overline{\partial} g = \overline{\partial}(g \circ f^{-1} \circ f) &= \big(\partial(g \circ f^{-1}) \circ f\big) \cdot \overline{\partial} f + \big(\overline{\partial}(g \circ f^{-1}) \circ f\big) \cdot \overline{\partial f}, \\
\partial g = \partial(g \circ f^{-1} \circ f) &= \big(\partial(g \circ f^{-1}) \circ f\big) \cdot \partial f + \big(\overline{\partial}(g \circ f^{-1}) \circ f\big) \cdot \overline{\overline{\partial} f}.
\end{aligned}$$

These two equations imply the following chain of equations and implications:

$$\frac{\overline{\partial} g}{(\partial(g \circ f^{-1}) \circ f)} = \frac{\mu(g) \cdot \partial g}{(\partial(g \circ f^{-1}) \circ f)}$$

$$\Longrightarrow$$

$$\overline{\partial} f + \big(\mu(g \circ f^{-1}) \circ f\big) \cdot \overline{\partial f} = \mu(g) \cdot \partial f + \mu(g) \cdot \big(\mu(g \circ f^{-1}) \circ f\big) \cdot \overline{\overline{\partial} f}$$

$$\Longrightarrow$$

$$\mu(g \circ f^{-1}) \circ f = \frac{\mu(g) \cdot \partial f - \overline{\partial} f}{\overline{\partial f} - \mu(g) \cdot \overline{\overline{\partial} f}} = \frac{\mu(g) - \mu(f)}{\frac{\overline{\partial f}}{\partial f} - \mu(g) \cdot \frac{\overline{\overline{\partial} f}}{\partial f}} = \frac{\partial f}{\overline{\partial f}} \frac{\mu(g) - \mu(f)}{1 - \mu(g)\overline{\mu(f)}}.$$

Note that the quotients in all these equations are well-defined almost everywhere. ☐

A *Beltrami coefficient on $D \subset \mathbb{C}$* is a bounded measurable map $\mu \colon D \to \mathbb{C}$. We will mostly need the case in which μ has norm strictly less than 1. D will typically be the upper half plane or the entire plane. Recall the famous *measurable Riemann mapping theorem* stated below. A proof of this can be found in [9, p. 102].

Theorem 2.2. *Given a bounded measurable function $\mu \colon \mathbb{C} \to \mathbb{C}$ of supremums norm strictly less than 1, there exists a homeomorphism f of the Riemann sphere that is a quasi-conformal map with coefficient μ when restricted to $\mathbb{C}$, meaning that $\overline{\partial} f = \mu \cdot \partial f$. The map f is uniquely determined by the condition that it fixes 0, 1, and ∞.*

We can easily deduce a version of the measurable Riemann mapping theorem for the upper half plane.

Corollary 2.3. *Given a bounded measurable function $\mu \colon \mathbb{H} \to \mathbb{C}$ of supremums norm strictly less than 1, there exists a homeomorphism f of the closed upper half plane that is a quasi-conformal map with coefficient μ when restricted to $\mathbb{H}$, meaning that $\overline{\partial} f = \mu \cdot \partial f$. The map f is uniquely determined by the condition that it fixes 0, 1, and ∞.*

Proof. Given such μ, we define a new function on $\mathbb{C}$ by setting

$$\nu(z) = \begin{cases} \mu(z), & \text{if } z \in \mathbb{H} \\ 0, & \text{if } z \in \mathbb{R} \\ \overline{\mu(\overline{z})}, & \text{if } z \in \mathbb{H}^*. \end{cases}$$

The measurable Riemann mapping theorem yields a unique homeomorphism $f\colon \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ with coefficient ν fixing 0, 1, and ∞. Using the chain and conjugation rule for the Wirtinger derivatives, one easily verifies that the map $z \mapsto \overline{f(\overline{z})}$ also has coefficient ν. Thus, by uniqueness of f, we have $f(z) = \overline{f(\overline{z})}$ and, in particular, $f(\mathbb{R}) = \mathbb{R}$. As a quasi-conformal map, f is orientation-preserving, which guarantees us that $f(\mathbb{H}) = \mathbb{H}$. Restricting f to the closed upper half plane shows existence of the desired solution. Uniqueness can be proved in a similar fashion. Suppose g is another solution. Then the function

$$\begin{cases} g(z), & \text{if } z \in \overline{\mathbb{H}} \\ \overline{g(\overline{z})}, & \text{if } z \in \mathbb{H}^* \end{cases}$$

has coefficient ν and is equal to f by uniqueness. □

Later on, we will need that quasi-conformal maps on the upper half plane (and, hence, any conformally equivalent domain) with image a Jordan domain, i.e. a domain that is bounded by a Jordan curve, can be extended to a homeomorphism between the closures, by composing with a conformal map.

Proposition 2.4. *Given a quasi-conformal map $f\colon \mathbb{H} \to D$, where D is a Jordan domain, there exists a homeomorphism $\overline{f}\colon \overline{\mathbb{H}} \to \overline{D}$ that is conformally equivalent to f on $\mathbb{H}$.*

Proof. Let $\mu(f)$ be the Beltrami coefficient of f, and set

$$\mu(z) = \begin{cases} \mu(f)(z), & \text{if } z \in \mathbb{H} \\ 0, & \text{if } z \in \overline{\mathbb{H}^*}. \end{cases}$$

The measurable Riemann mapping theorem provides us with a homeomorphism f_μ of $\overline{\mathbb{C}}$ with coefficient μ fixing 0, 1, and ∞. Set $g = f_\mu \circ f^{-1}\colon D \to f_\mu(\mathbb{H})$. This map is conformal inside D, by construction. By Carathéodory's theorem, g has a continuous extension to a homeomorphism $\overline{g}\colon \overline{D} \to \overline{f_\mu(\mathbb{H})}$. Then the restriction of $\overline{g}^{-1} \circ f_\mu$ to $\overline{\mathbb{H}}$ is the desired extension of f. □

We will be interested in specific Beltrami coefficients that preserve Γ in some sense. We call $\mathrm{BC}(\Gamma)$ the vector space of Beltrami coefficients $\mu\colon \mathbb{H} \to \mathbb{C}$ that satisfy the following property: for all $\gamma \in \Gamma$ we require

$$\mu = (\mu \circ \gamma)\,\frac{\overline{\gamma'}}{\gamma'}.$$

Here, γ' denotes the derivative in the usual sense, i.e. $\partial\gamma$. We denote by $\mathrm{BC}(\Gamma)_1$ the set of elements $\mu \in \mathrm{BC}(\Gamma)$ that have supremums norm strictly bounded by 1. By the measurable Riemann mapping theorem for $\mathbb{H}$, every element $\mu \in \mathrm{BC}(\Gamma)_1$ admits a unique solution as specified in corollary 2.3. We denote this solution by $w^\mu\colon \overline{\mathbb{H}} \to \overline{\mathbb{H}}$. The space of these Beltrami solutions will be denoted by

$$\mathrm{BS}(\Gamma) = \left\{ w^\mu \;\middle|\; \mu \in \mathrm{BC}(\Gamma)_1 \right\}.$$

We define an equivalence relation on $\mathrm{BS}(\Gamma)$ by saying $w^\mu \sim w^\nu$ if w^μ and w^ν agree on the real line. The quotient space will be denoted by

$$\mathrm{TS}'(\Gamma) = \left\{ [w^\mu] \;\middle|\; \mu \in \mathrm{BS}(\Gamma) \right\}.$$

The choice of naming it $\mathrm{TS}'(\Gamma)$ will soon be motivated when we construct a bijection between this set and some Teichmüller space. Let us proceed with more definitions. We write $\mathrm{QC}(\Gamma)$ for the set of quasi-conformal maps from $\mathbb{H}/\Gamma$ to some other Riemann surface,

$$\mathrm{QC}(\Gamma) = \left\{ f \;\middle|\; f\colon \mathbb{H}/\Gamma \to \mathbb{H}/\Gamma_f \text{ is quasi-conformal, where } \Gamma_f \text{ is some Fuchsian group} \right\}.$$

We declare two elements $f, g \in \mathrm{QC}(\Gamma)$ to be equivalent if the composition $f \circ g^{-1}\colon \mathbb{H}/\Gamma_g \to \mathbb{H}/\Gamma_f$ is isotopic to a conformal map. The *Teichmüller space of* Γ is defined as the set of equivalence classes

$$\mathrm{Teich}(\Gamma) = \left\{ [f] \;\middle|\; f \in \mathrm{QC}(\Gamma) \right\}.$$

Remark 2.5. *As discussed in chapter 1.1, there is a natural identification of* $\mathrm{Teich}(\mathcal{S}_g)$ *with* $\mathrm{Teich}(\Gamma)$*.*

Our next goal is to construct a bijection between $\mathrm{Teich}(\Gamma)$ and $\mathrm{TS}'(\Gamma)$. First, we define a map from $\mathrm{QC}(\Gamma)$ to $\mathrm{BS}(\Gamma)$ as follows. Given $f \in \mathrm{QC}(\Gamma)$, there exists a lift $f^*\colon \mathbb{H} \to \mathbb{H}$ of f that is a quasi-conformal map itself. By proposition 2.4, there is an extension $\overline{f^*}\colon \overline{\mathbb{H}} \to \overline{\mathbb{H}}$ of f^* that is a homeomorphism and is conformally equivalent to f^* on $\mathbb{H}$. Moreover, by composing it with a Möbius transformation, we can take $\overline{f^*}$ to have fixed points 0, 1, and ∞. This way, $\overline{f^*}$ is a solution for the Beltrami coefficient $\mu(f^*)$ of f^* in the sense of corollary 2.3, and, therefore, it is uniquely determined. If we show that $\mu(f^*)$ has the property

$$\mu(f^*) = (\mu(f^*) \circ \gamma)\, \frac{\overline{\gamma'}}{\gamma'}$$

for every $\gamma \in \Gamma$, then $\overline{f^*}$ is an element of $\mathrm{BS}(\Gamma)$.

Remark 2.6. *We call* $\overline{f^*}$ *as described above an "extension" even though it does not necessarily agree with* f^* *on* $\mathbb{H}$*. In this context, we use the word "extension" when the two maps are conformally equivalent on* $\mathbb{H}$*, i.e. agree up to composition with a conformal map.*

Lemma 2.7. *The assignment* $f \mapsto \overline{f^*}$ *is a well-defined map from* $\mathrm{QC}(\Gamma)$ *to* $\mathrm{BC}(\Gamma)$*. Moreover, it is surjective.*

Proof. First note that, due to the equivariance condition, the composition $f^* \circ \gamma \circ (f^*)^{-1}$ is again a Möbius transformation and, in particular, holomorphic. To shorten notation, denote this composition by g. With this notation, we have $g \circ f^* = f^* \circ \gamma$. Separately taking the derivatives ∂ and $\overline{\partial}$ on both sides and using the chain rule for the Wirtinger derivatives yields the two equations

$$\begin{aligned}(\partial g \circ f^*) \cdot \partial f^* &= (\partial f^* \circ \gamma) \cdot \gamma' \\ (\partial g \circ f^*) \cdot \overline{\partial} f^* &= (\overline{\partial} f^* \circ \gamma) \cdot \overline{\gamma'}.\end{aligned}$$

Dividing the second line by the first is defined almost everywhere and equals

$$\mu(f^*) = (\mu(f^*) \circ \gamma)\, \frac{\overline{\gamma'}}{\gamma'},$$

as desired. Hence,

$$\mathrm{QC}(\Gamma) \to \mathrm{BS}(\Gamma),\;\; f \mapsto \overline{f^*}$$

is well-defined. Given an element $w^\mu \in \mathrm{BS}(\Gamma)$, $\mu \in \mathrm{BC}(\Gamma)_1$, we can perform the previous calculation in reverse. Again, to shorten notation, we set $g = w^\mu \circ \gamma \circ (w^\mu)^{-1}$. As before, we separately take the derivatives ∂ and $\overline{\partial}$ on both sides to get the two equations

$$\begin{aligned}(\partial g \circ w^\mu) \cdot \partial w^\mu + (\overline{\partial} g \circ w^\mu) \cdot \partial \overline{w^\mu} &= (\partial w^\mu \circ \gamma) \cdot \gamma' \\ (\partial g \circ w^\mu) \cdot \overline{\partial} w^\mu + (\overline{\partial} g \circ w^\mu) \cdot \overline{\partial}\, \overline{w^\mu} &= (\overline{\partial} w^\mu \circ \gamma) \cdot \overline{\gamma'}.\end{aligned}$$

Dividing the second line by the first almost everywhere gives rise to

$$\frac{(\partial g \circ w^\mu) \cdot \overline{\partial} w^\mu + (\overline{\partial} g \circ w^\mu) \cdot \overline{\partial} \overline{w^\mu}}{(\partial g \circ w^\mu) \cdot \partial w^\mu + (\overline{\partial} g \circ w^\mu) \cdot \partial \overline{w^\mu}} = (\mu \circ \gamma) \frac{\overline{\gamma'}}{\gamma'} = \mu.$$

Thus, we see that $g = w^\mu \circ \gamma \circ (w^\mu)^{-1}$ must actually be holomorphic. Since it is also a homeomorphism of $\overline{\mathbb{H}}$, it must be a Möbius transformation and, hence, the set $w^\mu \Gamma (w^\mu)^{-1}$ is a Fuchsian group. In particular, w^μ descends to a quasi-conformal map $\mathbb{H}/\Gamma \to \mathbb{H}/\Gamma'$, $\Gamma' = w^\mu \Gamma (w^\mu)^{-1}$, which shows that the map $\mathrm{QC}(\Gamma) \to \mathrm{BS}(\Gamma)$ is surjective. □

That we can identify $\mathrm{Teich}(\Gamma)$ and $\mathrm{TS}'(\Gamma)$ follows from the next proposition.

Proposition 2.8. *Given two elements $f, g \in \mathrm{QC}(\Gamma)$, the composition $f \circ g^{-1}$ is isotopic to a conformal map if and only if the induced maps $\overline{f^*}$ and $\overline{g^*}$ agree on the real line.*

Proof. Suppose first that $\overline{f^*}$ and $\overline{g^*}$ agree on the real line. To shorten notation, let us abbreviate $g_0 = f \circ g^{-1} \colon \mathbb{H}/\Gamma_g \to \mathbb{H}/\Gamma_f$. Let g_0^* be a lift of g_0 and let $[g_0]_*$ denote the induced map $[g_0]_* \colon \Gamma_g \to \Gamma_f$. For $\gamma \in \Gamma_g$, the equivariance condition

$$g_0^* \circ \gamma = [g_0]_*(\gamma) \circ g_0^*$$

holds. Next, extend the maps to the boundary such that the extensions fix 0, 1, and ∞. Let us write $\overline{g_0^*}$ for the extension of g_0^*. On $\mathbb{H}$, the map $\overline{g_0^*}$ is conformally equivalent to g_0^*, i.e. $h^* \circ \overline{g_0^*} = g_0^*$ for some conformal map h^*. Then

$$h^* \circ \overline{g_0^*} \circ \gamma = [g_0]_*(\gamma) \circ h^* \circ \overline{g_0^*}$$

for all $\gamma \in \Gamma_g$. Note that because the lift $(f \circ g^{-1})^*$ is conformally equivalent to $f^* \circ (g^*)^{-1}$ and because the lifts are taken so that they fix 0, 1, and ∞, the extension $\overline{g_0^*}$ is the same map as $\overline{f^*} \circ (\overline{g^*})^{-1}$. In particular, the hypothesis state that $\overline{g_0^*}$ is the identity on $\mathbb{R}$. Thus, using $\gamma(\partial\mathbb{H}) = \partial\mathbb{H}$, we have

$$h^* \circ \gamma = h^* \circ \overline{g_0^*} \circ \gamma = [g_0]_*(\gamma) \circ h^* \circ \overline{g_0^*} = [g_0]_*(\gamma) \circ h^*$$

on the boundary of $\mathbb{H}$, for all $\gamma \in \Gamma_g$. This shows that h^* descends to a conformal map $h \colon \mathbb{H}/\Gamma_g \to \mathbb{H}/\Gamma_f$ because all these maps are Möbius transformations and equality on $\partial\mathbb{H}$ implies equality on $\mathbb{H}$. In order to conclude the first direction of the equivalence, we show that g_0 is isotopic to h. Observe that if c is the geodesic between $g_0^*(z)$ and $h^*(z)$, $z \in \mathbb{H}$, then $[g_0]_*(\gamma)(c)$ is the geodesic between $[g_0]_*(\gamma) \circ g_0^*(z) = g_0^* \circ \gamma(z)$ and $[g_0]_*(\gamma) \circ h^*(z) = h^* \circ \gamma(z)$. Hence, if we denote the geodesic between $g_0^*(z)$ and $h^*(z)$ by $G_t^*(z) = c(t)$, then G_t^* factors through Γ_g,

$$G_t^* \circ \gamma(z) = [g_0]_*(\gamma) \circ G_t^*(z),$$

and, hence, descends to an isotopy $G_t \colon \mathbb{H}/\Gamma_g \to \mathbb{H}/\Gamma_f$ between g_0 and h. Next, suppose we know that $g_0 = f \circ g^{-1}$ is isotopic to some conformal map h. The isotopy enforces $[g_0]_* = [h]_*$. Indeed, suppose the isotopy is realized by the homeomorphisms G_t. Then the map $[G_t]_* \colon \Gamma_g \to \Gamma_f$ depends continuously on t and, therefore, must be the same for all t. In particular, $[g_0]_* = [G_0]_* = [G_1]_* = [h]_*$. Since we assumed in the beginning of this chapter that $\mathbb{H}/\Gamma$ is a compact Riemann surface, we can take a compact fundamental domain F for Γ_g in $\mathbb{H}$. Let g_0^* and h^* be lifts of g_0 and h, respectively. Given $z' \in \mathbb{H}$, the equality $[g_0]_* = [h]_*$ assures that, for $\gamma \in \Gamma_g$ with $\gamma(z) = z'$, $z \in F$, we have

$$d(g_0^*(z'), h^*(z')) = d(g_0^* \circ \gamma(z), h^* \circ \gamma(z)) = d([h]_*(\gamma) \circ g_0^*(z), [h]_*(\gamma) \circ h^*(z)) = d(g_0^*(z), h^*(z)),$$

where the last equality is due to $[h]_*(\gamma) \in \Gamma_f$ being an isometry. Using compactness of F, we conclude that the distance between g_0^* and h^* is uniformly bounded in $\mathbb{H}$. Let h' realize the conformal equivalence on $\mathbb{H}$ between g_0^* and $\overline{g_0^*}$. Then the distance between $h' \circ \overline{g_0^*}$ and h^* is uniformly bounded in $\mathbb{H}$. Due to the nature of the hyperbolic metric, it follows that we must have $h' \circ \overline{g_0^*} = h^*$ on the boundary $\partial\mathbb{H}$. Thus, $\overline{g_0^*}$ agrees on $\partial\mathbb{H}$ with a Möbius transformation, and it fixes 0, 1, and ∞. Therefore, $\overline{f^*} \circ (\overline{g^*})^{-1} = \overline{g_0^*}$ is the identity on $\mathbb{R}$. $\square$

The forward direction of this equivalence implies that the map $\mathrm{QC}(\Gamma) \to \mathrm{BS}(\Gamma)$ descends to a well-defined map on the quotients,

$$\mathrm{Teich}(\Gamma) \to \mathrm{TS}'(\Gamma),\ [f] \mapsto [\overline{f^*}].$$

We already verified that this is a surjection. The backwards direction of the equivalence guarantees that this map also is injective. We conclude the the above map defines an identification of $\mathrm{Teich}(\Gamma)$ with $\mathrm{TS}'(\Gamma)$. However, we are actually interested in a slightly different set. Given a Beltrami coefficient $\mu \in \mathrm{BC}(\Gamma)_1$, define a coefficient on the entire plane by

$$\mu^*(z) = \begin{cases} \mu(z), & \text{if } z \in \mathbb{H} \\ 0, & \text{if } z \in \overline{\mathbb{H}^*}. \end{cases}$$

The measurable Riemann mapping theorem provides us with a unique solution, which we denote by $w_\mu \colon \overline{\mathbb{C}} \to \overline{\mathbb{C}}$, that fixes 0, 1, and ∞. We declare two such solutions w_μ and w_ν to be equivalent if they agree on the lower half plane $\mathbb{H}^*$. We denote the set of equivalence classes by

$$\mathrm{TS}(\Gamma) = \left\{ [w_\mu] \;\middle|\; \mu \in \mathrm{BC}(\Gamma)_1 \right\}.$$

The next lemma gives us a canonical identification of $\mathrm{TS}(\Gamma)$ and $\mathrm{TS}'(\Gamma)$.

Lemma 2.9. *Given $\mu, \nu \in \mathrm{BC}(\Gamma)_1$, w^μ and w^ν agree on $\mathbb{R}$ if and only if w_μ and w_ν agree on $\mathbb{H}^*$.*

Proof. First recall that w^μ is the restriction to $\overline{\mathbb{H}}$ of the solution to

$$\begin{cases} \mu(z), & \text{if } z \in \mathbb{H} \\ 0, & \text{if } z \in \mathbb{R} \\ \overline{\mu(\overline{z})}, & \text{if } z \in \mathbb{H}^*, \end{cases}$$

and w_μ is the solution to

$$\begin{cases} \mu(z), & \text{if } z \in \mathbb{H} \\ 0, & \text{if } z \in \overline{\mathbb{H}^*}. \end{cases}$$

In particular, the restrictions $w_\mu \circ (w^\mu)^{-1}\big|_{\mathbb{H}}$, $w^\nu \circ (w_\nu)^{-1}\big|_{\mathbb{H}}$ and $w_\mu \circ (w_\nu)^{-1}\big|_{\overline{\mathbb{H}^*}}$ are all conformal. If w^μ and w^ν agree on $\mathbb{R}$, then

$$\begin{cases} (w^\mu)^{-1} \circ w^\nu(z), & \text{if } z \in \overline{\mathbb{H}} \\ z, & \text{if } z \in \mathbb{H}^* \end{cases}$$

is a quasi-conformal map, and, hence,

$$f(z) = \begin{cases} w_\mu \circ (w^\mu)^{-1} \circ w^\nu \circ (w_\nu)^{-1}(z), & \text{if } z \in \overline{\mathbb{H}} \\ w_\mu \circ (w_\nu)^{-1}(z), & \text{if } z \in \mathbb{H}^* \end{cases}$$

is conformal. Since each map in the definition of f has fixed points 0, 1, and ∞, so does f. Hence, f is the identity. In particular, $w_\mu = w_\nu$ on $\mathbb{H}^*$. Conversely, if $w_\mu = w_\nu$ on $\mathbb{H}^*$, then the restriction of $(w_\mu)^{-1} \circ w_\nu$ to $\overline{\mathbb{H}}$ takes values in $\overline{\mathbb{H}}$. Thus, the map

$$g\colon \overline{\mathbb{H}} \to \overline{\mathbb{H}},\ z \mapsto w^\mu \circ (w_\mu)^{-1} \circ w_\nu \circ (w^\nu)^{-1}$$

is well-defined. Moreover, it is conformal by an observation similar to the one above. As before, g fixes 0, 1, and ∞, and, therefore, it must be the identity. Since $w_\mu = w_\nu$ on $\mathbb{R}$ by continuity, we can conclude that $w^\mu = w^\nu$ on $\mathbb{R}$. □

For further discussions, we identify Teich(Γ) with TS(Γ). Now suppose that we are given a conformal map f defined on some domain in $\mathbb{C}$ with image in $\mathbb{C}$. We define the *Schwarzian derivative* of f at a point z as

$$s(f)(z) = \frac{f'''(z)}{f'(z)} - \frac{3}{2}\left(\frac{f''(z)}{f'(z)}\right)^2.$$

This definition is motivated by the fact that $s(f)$ vanishes identically if f is any Möbius transformation. In fact, f is a Möbius transformation if and only if $s(f)$ vanishes identically. This follows from solving the differential equation $s(f) = 0$. Given a Beltrami coefficient $\mu \in \mathrm{BC}(\Gamma)_1$, set

$$S_\mu\colon \mathbb{H}^* \to \mathbb{C},\ z \mapsto s(w_\mu)(z).$$

This map has the following properties.

Proposition 2.10. *For $\gamma \in \Gamma$, we have on $\mathbb{H}^*$:*

$$(S_\mu \circ \gamma) \cdot (\gamma')^2 = S_\mu.$$

Moreover, we have $[w_\mu] = [w_\nu]$ in Teich(Γ) *if and only if $S_\mu = S_\nu$ on $\mathbb{H}^*$.*

Proof. A direct computation shows that, for any two conformal maps f and g and for all points z for which $g(z)$ is in the domain of f, we have

$$s(f \circ g)(z) = (s(f) \circ g)(z) \cdot g'(z)^2 + s(g)(z).$$

Since $\gamma_\mu = w_\mu \circ \gamma \circ (w_\mu)^{-1}$ is a Möbius transformation, we get

$$(s(w_\mu) \circ \gamma) \cdot (\gamma')^2 + \underbrace{s(\gamma)}_{=0} = s(w_\mu \circ \gamma) = s(\gamma_\mu \circ w_\mu) = (\underbrace{s(\gamma_\mu)}_{=0} \circ w_\mu) \cdot (w_\mu')^2 + s(w_\mu) = s(w_\mu).$$

This proves the first statement. The assertion that $[w_\mu] = [w_\nu]$ implies $S_\mu = S_\nu$ on $\mathbb{H}^*$ is trivial. Now suppose that $S_\mu = S_\nu$ on $\mathbb{H}^*$. Define a new conformal map by

$$f\colon w_\mu(\mathbb{H}^*) \to w_\nu(\mathbb{H}^*),\ z \mapsto w_\nu \circ (w_\mu)^{-1}(z).$$

Using the chain rule for the Schwarzian derivative once more, we see that

$$s(w_\nu) = s(f \circ w_\mu) = (s(f) \circ w_\mu) \cdot (w_\mu')^2 + \underbrace{s(w_\mu)}_{=s(w_\nu)}$$

on $\mathbb{H}^*$. In order for this equality to hold, we must have $s(f) \equiv 0$ on $w_\mu(\mathbb{H}^*)$. As remarked before the lemma, this implies that f must be a Möbius transformation. Since w_μ and w_ν have 0, 1, and ∞ as fixed points, so does f, and, therefore, f must be the identity. In particular, we have $w_\mu = w_\nu$ on $\mathbb{H}^*$. □

The first assertion of this proposition tells us that S_μ defines a *holomorphic quadratic differential*[7] on the Riemann surface $\mathbb{H}^*/\Gamma$. By the second statement, we get an injection

$$\mathrm{Teich}(\Gamma) \to \mathrm{QD}^*(\Gamma),\ [w_\mu] \mapsto S_\mu,$$

where $\mathrm{QD}^*(\Gamma)$ denotes the complex vector space of holomorphic quadratic differentials on $\mathbb{H}^*/\Gamma$,

$$\mathrm{QD}^*(\Gamma) = \left\{\phi\colon \mathbb{H}^* \to \mathbb{C} \;\middle|\; \phi \text{ is holomorphic and } (\phi \circ \gamma)\cdot(\gamma')^2 = \phi \text{ on } \mathbb{H}^*, \text{ for all } \gamma \in \Gamma\right\}.$$

Remark 2.11. *There is a natural bijection between* $\mathrm{QD}^*(\Gamma)$ *and the analogously defined* $\mathrm{QD}(\Gamma)$ *(where* $\mathbb{H}^*$ *is replaced by* $\mathbb{H}$*) given by* $\phi(z) \leftrightarrow \overline{\phi(\overline{z})}$*.*

Proof. That $\overline{\phi(\overline{z})}$ is holomorphic follows easily from the chain rule for $\overline{\partial}$, and that it is an element of $\mathrm{QD}(\Gamma)$ is due to $\gamma(\overline{z}) = \overline{\gamma(z)}$ and $\gamma'(\overline{z}) = \overline{\gamma'(z)}$ for Möbius transformations. □

The next lemma is purely analytic and will not be proved here. The reader may consult [9, p. 108]. We equip the vector space $\mathrm{BC}(\Gamma)$ with the supremums norm.

Lemma 2.12. *If* $(\mu_n)_{n\geq 0} \subset \mathrm{BC}(\Gamma)_1$ *converges to 0 in* $\mathrm{BC}(\Gamma)$*, then* $(w_{\mu_n})_{n\geq 0}$ *converges to the identity map uniformly on compact sets.*

In a nutshell, if μ_n converges to 0 in $\mathrm{BC}(\Gamma)$, then $\overline{\partial} w_{\mu_n}$ converges to 0 in the supremums norm, which implies that w_{μ_n} converges to a conformal map. This conformal map must be the identity since each w_{μ_n} fixes 0, 1, and ∞. The analytic bit of the proof is to show that the convergence is uniform on compact sets. Using this, we can map Teichmüller space not only injectively, but continuously into $\mathrm{QD}^*(\Gamma)$.

Lemma 2.13. *The two maps*

$$\begin{aligned} \mathrm{BC}(\Gamma)_1 &\to \mathrm{QD}^*(\Gamma), & \mu &\mapsto S_\mu \\ \mathrm{Teich}(\Gamma) &\to \mathrm{QD}^*(\Gamma), & [w_\mu] &\mapsto S_\mu \end{aligned}$$

are both continuous.

Proof. Continuity of the second map follows from continuity of the first since $\mathrm{Teich}(\Gamma) \simeq \mathrm{TS}(\Gamma)$ inherits the quotient topology of $\mathrm{BC}(\Gamma)_1$. We prove below that $\mathrm{QD}^*(\Gamma)$ is a finite dimensional vector space. Thus, any norm induces the correct topology and we might as well use the norm given by

$$||\phi||_\infty = \sup_{z\in\mathbb{H}^*} \Im(z)^2|\phi(z)|,$$

where $\Im$ is the imaginary part. This way, a sequence $(\phi_n)_{n\geq 0} \subset \mathrm{QD}^*(\Gamma)$ converges to ϕ in $\mathrm{QD}^*(\Gamma)$ if and only if $(\phi_n)_{n\geq 0}$ converges to ϕ uniformly on compact sets in $\mathbb{H}^*$. To see this, the reader simply needs to verify the equation

$$\Im(\gamma(z))^2 = \Im(z)^2\gamma'(z)\gamma'(\overline{z}),$$

because then we can fix any compact fundamental domain $F \subset \mathbb{H}^*$ of Γ and get

$$||\phi||_\infty = \sup_{z\in\mathbb{H}^*} \Im(z)^2|\phi(z)| = \sup_{z\in F} \Im(z)^2|\phi(z)|.$$

Therefore, continuity of the first map is a corollary of the previous lemma 2.12. Indeed, if $(\mu_n)_{n\geq 0}$ converges to μ in $\mathrm{BC}(\Gamma)$, then lemma 2.1 tells us that the Beltrami coefficient $\mu(w_{\mu_n} \circ w_\mu^{-1})$ converges to 0 in $\mathrm{BC}(\Gamma)$. Hence, by the last lemma, $w_{\mu_n} \circ w_\mu^{-1}$ converges to the identify uniformly on compact sets. □

[7]Usually, this is called a *holomorphic automorphic form of weight* -4 instead of a holomorphic quadratic differential. However, these are essentially the same objects considered from two different points of view. In the first case, we consider Fuchsian models, and, in the other case, we consider complex charts; compare this to the holomorphic quadratic differentials introduced in chapter 1.1.

Note that $\mathrm{QD}^*(\Gamma)$ is a complex vector space of real dimension $6g-6$ (see the proof of corollary 2.14 below). Since $\mathrm{Teich}(\Gamma)$ is homeomorphic to $\mathbb{R}^{3g-3}_{>0} \times \mathbb{R}^{3g-3}$, the Invariance of Domain theorem implies that the image of $\mathrm{Teich}(\Gamma)$ in $\mathrm{QD}^*(\Gamma)$ is open and that the continuous injection above is, in fact, an embedding. We identify $\mathrm{Teich}(\Gamma)$ with its image in $\mathrm{QD}^*(\Gamma)$. Thus, we get the following corollary.

Corollary 2.14. *The space* $\mathrm{Teich}(\Gamma)$ *has a complex manifold structure.*

Sketch of Proof. The proof of this corollary is reduced to proving that the vector space $\mathrm{QD}^*(\Gamma)$ has real dimension $6g-6$. Firstly, we recall that an advanced version of the measurable Riemann mapping theorem asserts that the solutions of Beltrami coefficients μ_t vary analytically in t when the coefficients μ_t themselves vary analytically in t, see [1, p. 390]. Moreover, the measurable Riemann mapping theorem gives rise to a map from the unit ball in $\mathrm{QD}^*(\Gamma)$ to $\mathrm{Teich}(\Gamma)$, which is given by

$$\mathrm{QD}^*(\Gamma)_1 \to \mathrm{Teich}(\Gamma) \subset \mathrm{QD}^*(\Gamma),\ q \mapsto [h],$$

where h is the solution of the coefficient specified in Teichmüller's theorem 1.4.[8] The uniqueness statement of that theorem tells us that this map is injective. The remark about analytic dependence of such a solution justifies that this map is analytic and, in particular, continuous. Thus, if we can show that $\mathrm{QD}^*(\Gamma)$ has real dimension at least $6g-6$, then the Invariance of Domain theorem asserts that the above map is a homeomorphism onto its image and, hence, that $\mathrm{QD}^*(\Gamma)$ has real dimension exactly equal to $6g-6$. Therefore, we make the following claim: $\mathrm{QD}^*(\Gamma)$ has real dimension at least $6g-6$. Take an element $\phi_0 \in \mathrm{QD}^*(\Gamma)$ with only simple zeros. In short, it follows from the existence of natural coordinates (cf. chapter 1.1) and the *Euler-Poincaré formula* for foliations that any holomorphic quadratic differential has exactly $4g-4$ zeros counted with multiplicity, see [6, p. 301,310-312]. Since for ϕ_0 the multiplicity is always 1, ϕ_0 has exactly $4g-4$ different zeros. Denote the set of these zeros by P. Let D_P be the vector space of meromorphic functions on $\mathbb{H}/\Gamma$ whose poles lie in P and are at most simple. By the famous *Riemann-Roch theorem* (see, for instance, [7, p. 101]), the (real) dimension of D_P is at least $|P|+1-g = 3g-3$. All that remains is to observe that the map

$$\mathrm{QD}^*(\Gamma) \to D_P,\ \phi \mapsto \frac{\phi}{\phi_0}$$

clearly is a vector space isomorphism. □

Let us now investigate the tangent bundle of $\mathrm{Teich}(\Gamma)$. Since we regarded $\mathrm{Teich}(\Gamma)$ as its embedded image in $\mathrm{QD}^*(\Gamma)$, which is a vector space, the tangent space of $\mathrm{Teich}(\Gamma)$ at the base point $0 \simeq [\mathrm{id}_{\overline{\mathbb{C}}}]$ is identified with $\mathrm{QD}^*(\Gamma) \simeq T_0(\mathrm{Teich}(\Gamma))$. Given an element $\phi \in \mathrm{QD}^*(\Gamma)$, we get an element in $\mathrm{BC}(\Gamma)$ defined by

$$H[\phi](z) = \Im(z)^2\phi(\overline{z}).$$

That this really is an element in $\mathrm{BC}(\Gamma)$ follows from the equality

$$\Im(\gamma(z))^2 = \Im(z)^2\gamma'(z)\gamma'(\overline{z})$$

for Möbius transformations, which we already used before. We call the image $H(\mathrm{QD}^*(\Gamma)) \subset \mathrm{BC}(\Gamma)$ the (vector) space of *harmonic Beltrami coefficients* and denote it by $\mathrm{HBC}(\Gamma)$. Clearly, H is injective and linear, and, therefore, we get an isomorphism

$$T_0(\mathrm{Teich}(\Gamma)) \simeq \mathrm{QD}^*(\Gamma) \simeq \mathrm{HBC}(\Gamma).$$

[8] To make sense of this, note that the notion of a holomorphic quadratic differential from chapter 1.1 is identical with the definition of $\mathrm{QD}(\Gamma)$, by using Fuchsian models for Riemann surfaces, and that Teichmüller's theorem has an analogue in the language of Fuchsian groups. Moreover, we can work in $\mathbb{H}^*$ instead of $\mathbb{H}$, by remark 2.11.

Remark 2.15. *Consider the bilinear pairing*

$$\mathrm{BC}(\Gamma) \times \mathrm{QD}^*(\Gamma) \to \mathbb{R},\ (\mu, \phi) \mapsto \int_F \mu(z)\overline{\phi(\overline{z})}\ dA(z),$$

where $F \subset \mathbb{H}$ is a fundamental domain for Γ. This is independent of the choice of F, by the transformation properties of μ and ϕ. We denote by $N(\Gamma) \subset \mathrm{BC}(\Gamma)$ the orthogonal complement of $\mathrm{QD}^*(\Gamma)$ with respect to this pairing, i.e. $N(\Gamma)$ is the kernel of the map

$$\Lambda\colon \mathrm{BC}(\Gamma) \to \mathrm{QD}^*(\Gamma)^*,\ \mu \mapsto (\mu, \cdot),$$

where $\mathrm{QD}^*(\Gamma)^*$ is the dual space of $\mathrm{QD}^*(\Gamma)$. The vector space $\mathrm{BC}(\Gamma)$ splits into the direct sum[9]

$$\mathrm{BC}(\Gamma) = \mathrm{HBC}(\Gamma) \oplus N(\Gamma).$$

In particular, we have

$$\mathrm{Image}(\Lambda) \simeq \mathrm{BC}(\Gamma)/N(\Gamma) \simeq \mathrm{HBC}(\Gamma) \simeq \mathrm{QD}^*(\Gamma).$$

Thus, the map Λ is actually surjective, and we get an identification of the cotangent space, given by

$$T_0^*(\mathrm{Teich}(\Gamma)) \simeq \mathrm{HBC}(\Gamma)^* \overbrace{\simeq}^{\textit{via } \Lambda} (\mathrm{QD}^*(\Gamma)^*)^* \simeq \mathrm{QD}^*(\Gamma).$$

So far, we have only classified the tangent space of $\mathrm{Teich}(\Gamma)$ at the base point. Of course, we also would like to have a description of the tangent space at a general point. Fix a point $p \in \mathrm{Teich}(\Gamma)$ with representative w_ν, and let Γ_ν denote the Fuchsian group $w_\nu \Gamma w_\nu^{-1}$. Given an element $[w_\mu] \in \mathrm{Teich}(\Gamma)$ with representative w_μ, the equivalence class $[w_\mu \circ w_\nu^{-1}]$ in $\mathrm{Teich}(\Gamma_\nu)$ is independent of the choice of representative of $[w_\mu]$ so that we get a well-defined map

$$p_*^\nu\colon \mathrm{Teich}(\Gamma) \to \mathrm{Teich}(\Gamma_\nu),\ [w_\mu] \mapsto [w_\mu \circ w_\nu^{-1}].$$

The map p_*^ν is biholomorphic, where we use the complex structures on $\mathrm{Teich}(\Gamma)$ and $\mathrm{Teich}(\Gamma_\nu)$ from corollary 2.14, i.e. $\mathrm{Teich}(\Gamma_\nu)$ inherits the complex structure from $\mathrm{QD}^*(\Gamma_\nu)$. Indeed, we mentioned before that an advanced version of the measurable Riemann mapping theorem states that the solutions of coefficients μ_t vary analytically in t if the coefficients μ_t vary analytically in t. Since $[w_\mu \circ w_\nu^{-1}]$ is the same element as $[w_\lambda]$, by lemma 2.1, where

$$\lambda = \left(\frac{\partial w_\nu}{\overline{\partial w_\nu}} \frac{\mu - \nu}{1 - \mu\overline{\nu}} \right) \circ w_\nu^{-1},$$

we see that an analytic perturbation of μ results in an analytic perturbation of λ and, hence, of $[w_\mu \circ w_\nu^{-1}]$. This shows that p_*^ν is holomorphic, and, since it clearly is bijective, the analytic Inverse Function Theorem implies that p_*^ν is biholomorphic. Observe that p_*^ν maps the point $p \in \mathrm{Teich}(\Gamma)$ to the base point in $\mathrm{Teich}(\Gamma_\nu)$. Thus, p_*^ν induces an isomorphism between $T_p(\mathrm{Teich}(\Gamma))$ and $T_0(\mathrm{Teich}(\Gamma_\nu))$. The latter can be identified with $\mathrm{HBC}(\Gamma_\nu)$, as discussed above. Thus, we got a complete description of the tangent bundle of $\mathrm{Teich}(\Gamma)$ via

$$p = [w_\nu] \implies T_p(\mathrm{Teich}(\Gamma)) \simeq \mathrm{HBC}(\Gamma_\nu),\ \Gamma_\nu = w_\nu \Gamma w_\nu^{-1}$$

and similarly for the cotangent bundle

$$p = [w_\nu] \implies T_p^*(\mathrm{Teich}(\Gamma)) \simeq \mathrm{QD}^*(\Gamma_\nu),\ \Gamma_\nu = w_\nu \Gamma w_\nu^{-1}.$$

Remark 2.16. $\mathrm{Teich}(\mathcal{S}_g)$ *inherits a complex manifold structure from the identification with* $\mathrm{Teich}(\Gamma)$. *That p_*^ν is biholomorphic shows that this complex structure is independent of the choice of Γ.*

[9] We postpone the proof of this to chapter 2.3 because it is not of direct interest for the current chapter.

2.2 The Weil-Petersson Metric

We begin by defining a positive inner product on the vector space $\mathrm{BC}(\Gamma)$. Fix a fundamental domain F of Γ in $\mathbb{H}$. Then we define the inner product of two elements $\mu, \nu \in \mathrm{BC}(\Gamma)$ by

$$(\mu, \nu)_{\mathrm{BC}(\Gamma)} = \int_F \frac{1}{\Im(z)^2} \mu(z) \overline{\nu(z)}\, dA(z),$$

where $dA(x+iy) = dxdy$ denotes the area element for integration. Similarly as before, this integral is independent of the choice of fundamental domain. We restrict this inner product to the harmonic Beltrami coefficients to get an inner product $(\cdot,\cdot)_0$ on $\mathrm{HBC}(\Gamma) \simeq T_0(\mathrm{Teich}(\Gamma))$. In the same way, we obtain an inner product $(\cdot,\cdot)_p$ on the tangent space $T_p(\mathrm{Teich}(\Gamma))$ from $(\cdot,\cdot)_{\mathrm{BC}(\Gamma_\nu)}$, where $\Gamma_\nu = w_\nu \Gamma w_\nu^{-1}$ for $p = [w_\nu]$. We can consider this collection of positive inner products on each tangent space as a positive symmetric inner product on the tangent bundle by taking the real part,

$$(g_{\mathrm{WP}})_p \colon T_p(\mathrm{Teich}(\Gamma)) \times T_p(\mathrm{Teich}(\Gamma)) \to \mathbb{R},\ (\mu, \nu) \mapsto 2\Re(\mu, \nu)_p,\ p \in \mathrm{Teich}(\Gamma).$$

This actually defines a Riemannian metric on $\mathrm{Teich}(\Gamma)$, see [9, p. 201].

Proposition 2.17. *The inner products $(\cdot,\cdot)_p$ depend smoothly on p.*

We call g_{WP} the *Weil-Petersson (Riemannian) metric on* $\mathrm{Teich}(\Gamma)$. It is compatible with the complex structure in the sense that

$$(g_{\mathrm{WP}})_p(i\mu, \nu) = 2\Re(i\mu, \nu)_p = -2\Re(\mu, i\nu)_p = -(g_{\mathrm{WP}})_p(\mu, i\nu) = -(g_{\mathrm{WP}})_p(i\nu, \mu).$$

Every compatible Riemannian metric on a complex manifold naturally induces a non-degenerate differential 2-form by setting

$$(\omega_{\mathrm{WP}})_p \colon T_p(\mathrm{Teich}(\Gamma)) \times T_p(\mathrm{Teich}(\Gamma)) \to \mathbb{R},\ (\mu, \nu) \mapsto (g_{\mathrm{WP}})_p(i\mu, \nu) = -2\Im(\mu, \nu)_p,\ p \in \mathrm{Teich}(\Gamma).$$

We call this differential form the *Weil-Petersson form on* $\mathrm{Teich}(\Gamma)$.

Theorem 2.18. *The Weil-Petersson form is closed. Therefore, since it is compatible with the complex structure by construction, it is Kähler.*

For a proof of this theorem, see [9, p. 202].The Weil-Petersson form has a very nice structure if we use Fenchel-Nielsen coordinates as a chart. This result is proved in [15, p. 976].

Theorem 2.19 (Wolpert's formula)**.** *Fix a pants decomposition of $\mathcal{S}_g$ for Fenchel-Nielsen coordinates l_j, θ_j, $1 \leq j \leq 3g-3$. Recall that θ_j was defined as the twist normalized by the length coordinate. We adjust the coordinates by $\tau_j = \frac{l_j}{2\pi}\theta_j$. Then ω_{WP} takes the form*

$$\omega_{\mathrm{WP}} = \sum_{1 \leq j \leq 3g-3} d\tau_j \wedge dl_j.$$

The proof of this theorem uses the following fact: if l_j and τ_j, $1 \leq j \leq 3g-3$, denote the adjusted Fenchel-Nielsen coordinates $\tau_j = \frac{l_j}{2\pi}\theta_j$, then $\frac{\partial}{\partial l_j}$ and $\frac{\partial}{\partial \tau_j}$, $1 \leq j \leq 3g-3$, form a basis of the tangent space of Teichmüller space. Further, if i denotes the complex structure on Teichmüller space, then one can show that the dual of $i\frac{\partial}{\partial \tau_j}$ is exactly dl_j, see [9, p. 227]. In other words, if

$$X(p) = \sum_{1 \leq j \leq 3g-3} a_j \frac{\partial}{\partial l_j}\bigg|_p + b_j \frac{\partial}{\partial \tau_j}\bigg|_p$$

is a tangent vector field on Teich(Γ), then the complex structure acts on it by

$$iX(p) = \sum_{1 \leq j \leq 3g-3} b_j \frac{\partial}{\partial l_j}\bigg|_p - a_j \frac{\partial}{\partial \tau_j}\bigg|_p .$$

The Weil-Petersson Riemannian metric on Teich(Γ) naturally induces a metric on Teich(Γ) by taking the infimum over lengths of connecting curves (just as any Riemannian metric on a manifold does). By identifying Teich(Γ) with Teich($\mathcal{S}_g$), we get a metric on Teich($\mathcal{S}_g$), which we denote by $\mathrm{d_{WP}}$ and call the *Weil-Petersson metric on* Teich($\mathcal{S}_g$). Let us verify that the topology induced by the Teichmüller metric is at least as fine as the one induced by $\mathrm{d_{WP}}$. First observe that the adjusted Fenchel-Nielsen coordinates give Teich($\mathcal{S}_g$) the same topology as the non-adjusted FN-coordinates. Given two points $[X],[Y] \in$ Teich($\mathcal{S}_g$), consider the linear path in adjusted FN-coordinates

$$\begin{aligned}\gamma(t) = \Big(& tL_{[X]}(\gamma_1) + (1-t)L_{[Y]}(\gamma_1), \dots, tL_{[X]}(\gamma_{3g-3}) + (1-t)L_{[Y]}(\gamma_{3g-3}), \\ & t\tau_1([X]) + (1-t)\tau_1([Y]), \dots, t\tau_{3g-3}([X]) + (1-t)\tau_{3g-3}([Y])\Big).\end{aligned}$$

Then

$$\begin{aligned}\gamma'(t) &= \sum_{1 \leq j \leq 3g-3} (L_{[X]}(\gamma_j) - L_{[Y]}(\gamma_j)) \frac{\partial}{\partial l_j}\bigg|_{\gamma(t)} + (\tau_j([X]) - \tau_j([Y])) \frac{\partial}{\partial \tau_j}\bigg|_{\gamma(t)}, \\ i\gamma'(t) &= \sum_{1 \leq j \leq 3g-3} (\tau_j([X]) - \tau_j([Y])) \frac{\partial}{\partial l_j}\bigg|_{\gamma(t)} - (L_{[X]}(\gamma_j) - L_{[Y]}(\gamma_j)) \frac{\partial}{\partial \tau_j}\bigg|_{\gamma(t)}\end{aligned}$$

and, hence,

$$\begin{aligned}(\omega_{\mathrm{WP}})_{\gamma(t)}(\gamma'(t), i\gamma'(t)) &= \sum_{1 \leq j \leq 3g-3} \Bigg(\underbrace{d\tau_j(\gamma'(t))}_{=\tau_j([X])-\tau_j([Y])} \overbrace{dl_j(i\gamma'(t))}^{=\tau_j([X])-\tau_j([Y])} - \underbrace{d\tau_j(i\gamma'(t))}_{=L_{[X]}(\gamma_j)-L_{[Y]}(\gamma_j)} \overbrace{dl_j(\gamma'(t))}^{=L_{[X]}(\gamma_j)-L_{[Y]}(\gamma_j)} \Bigg) = \\ &= \sum_{1 \leq j \leq 3g-3} (\tau_j([X]) - \tau_j([Y]))^2 + (L_{[X]}(\gamma_j) - L_{[Y]}(\gamma_j))^2\end{aligned}$$

Thus, if $[X]$ and $[Y]$ are δ-close in adjusted Fenchel-Nielsen coordinates, then

$$\begin{aligned}\mathrm{d_{WP}}([X],[Y]) &\leq \int_0^1 (\mathrm{g_{WP}})_{\gamma(t)}(\gamma'(t), \gamma'(t))\, dt = \int_0^1 (\omega_{\mathrm{WP}})_{\gamma(t)}(\gamma'(t), i\gamma'(t))\, dt \\ &= \sum_{1 \leq j \leq 3g-3} (\tau_j([X]) - \tau_j([Y]))^2 + (L_{[X]}(\gamma_j) - L_{[Y]}(\gamma_j))^2 \leq (3g-3)2\delta^2.\end{aligned}$$

We can also show that $\mathrm{d_{WP}}$ descends to a metric on Moduli space.

Proposition 2.20. *The Weil-Petersson metric is invariant under the action of* MCG($\mathcal{S}_g$) *on* Teich($\mathcal{S}_g$).

Sketch of Proof. We only need to observe what the action of MCG($\mathcal{S}_g$) is when using Teich(Γ) ≃ TS(Γ). Here, MCG($\mathcal{S}_g$) corresponds to the Modular group Mod(Γ) consisting of equivalence classes $[w]$ of quasi-conformal maps of the plane that satisfy $w\Gamma w^{-1} = \Gamma$, and $w_1 \sim w_2$ if $w_1 = \gamma \circ w_2$ for some $\gamma \in \Gamma$. Moreover, the action of $[w] \in \mathrm{Mod}(\Gamma)$ is given by

$$\mathrm{TS}(\Gamma) \to \mathrm{TS}(\Gamma),\ [w_\mu] \mapsto [\alpha \circ w_\mu \circ w^{-1}],$$

where α is a Möbius transformation such that the composition on the right hand side fixes 0, 1, and ∞. Thus, we see that this is the same transformation that we used at the end of last chapter to identify the other tangent spaces. From this observation and the definitions, it follows that g_{WP} is invariant under the action of $\mathrm{Mod}(\Gamma)$, and, hence, d_{WP} is invariant under the action of $\mathrm{MCG}(\mathcal{S}_g)$. $\square$

Thus, we can use proposition 1.12 to see that the Weil-Petersson metric also descends to Moduli space. By abuse of notation, we denote the Weil-Petersson metric on $\mathcal{M}(\mathcal{S}_g)$ also by

$$\mathrm{d}_{\mathrm{WP}}(X,Y) = \min_{[f]\in \mathrm{MCG}(\mathcal{S}_g)} \mathrm{d}_{\mathrm{WP}}([f]\cdot[X],[Y]).$$

Let us finish this chapter with collecting a few results from [12], [16] and [17], which we will need later on.

Theorem 2.21. *The augmented Teichmüller space is the metric completion of* $\mathrm{Teich}(\mathcal{S}_g)$ *with respect to the Weil-Petersson metric. Thus,* $\overline{\mathcal{M}(\mathcal{S}_g)}$ *also is the metric completion of* $\mathcal{M}(\mathcal{S}_g)$ *with respect to the (quotient-) Weil-Petersson metric.*

We will denote the complete Weil-Petersson metric on $\overline{\mathrm{Teich}(\mathcal{S}_g)}$ by $\mathrm{d}_{\overline{\mathrm{WP}}}$. In particular, this theorem states that $(\mathrm{Teich}(\mathcal{S}_g), \mathrm{d}_{\mathrm{WP}})$ is not a complete metric space. The Weil-Petersson metric induces the same topology on $\mathrm{Teich}(\mathcal{S}_g)$ as the Teichmüller metric. However, recall that the Teichmüller metric did not extend to $\overline{\mathrm{Teich}(\mathcal{S}_g)}$, especially not as its metric completion since it already is a complete metric on $\mathrm{Teich}(\mathcal{S}_g)$.

Theorem 2.22. *The two spaces* $(\mathrm{Teich}(\mathcal{S}_g), \mathrm{d}_{\mathrm{WP}})$ *and* $(\overline{\mathrm{Teich}(\mathcal{S}_g)}, \mathrm{d}_{\overline{\mathrm{WP}}})$ *are both geodesic spaces.*

We call a geodesic with respect to the Weil-Petersson metric a *WP-geodesic* to avoid confusion with geodesics in the Teichmüller metric.

Proposition 2.23. *The geodesic length functions are convex along Weil-Petersson geodesics, meaning that for any fixed geodesic γ in $\mathcal{S}_g$ and any WP-geodesic* $\{[X_t]\}_{t\in[0,T]} \subset \mathrm{Teich}(\mathcal{S}_g)$*, the map*

$$[0,T] \to \mathbb{R}_{\geq 0},\ t \mapsto L_{[X_t]}(\gamma)$$

is convex in the usual sense.

Next, fix a pants decomposition $\{\gamma_1, \ldots, \gamma_{3g-3}\}$ of $\mathcal{S}_g$. Let X denote a hyperbolic surface with nodes at every $\gamma_1, \ldots, \gamma_{3g-3}$. The equivalence class $[X]$ of this surface in $\overline{\mathrm{Teich}(\mathcal{S}_g)}$ is unique by remark 1.27.

Proposition 2.24. *The WP-distance from $[X]$ is locally given by*

$$\mathrm{d}_{\overline{\mathrm{WP}}}([X],[Y]) = \sqrt{2\pi L([Y])} + \mathcal{O}(L([Y])^2),$$

where $L([Y]) = L_{[Y]}(\gamma_1) + \cdots + L_{[Y]}(\gamma_{3g-3})$.

2.3 Tying Up Loose Ends

In this chapter, we mean to give two of the proofs that we left out earlier. We begin by proving the existence of the splitting $\mathrm{BC}(\Gamma) = \mathrm{HBC}(\Gamma) \oplus N(\Gamma)$ given in remark 2.15. To do so, we first need to develop a useful tool, called the *reproducing kernel*, and its applications. It is defined by

$$K\colon \mathbb{C}^2 \setminus \{\overline{z} = \zeta\} \to \mathbb{C},\ (z,\zeta) \mapsto \frac{12}{\pi} \frac{1}{(\overline{z} - \zeta)^4}.$$

Note that $K(z,\zeta) = \overline{K(\overline{z},\overline{\zeta})}$ and that, for any Möbius transformation γ, we have

$$K(z,\zeta) = K(\gamma(z),\gamma(\zeta))\ \overline{\gamma'(z)^2}\gamma'(\zeta)^2.$$

This follows from the fact that

$$\left(\frac{\gamma(\overline{z}) - \gamma(\zeta)}{\overline{z} - \zeta}\right)^2 = \gamma'(\overline{z})\gamma'(\zeta),$$

which we have already used several times. The formula for K together with the equivariance formula for an element $\mu \in \mathrm{BC}(\Gamma)$ establishes that the function

$$\Phi[\mu](z) = \int_{\mathbb{H}^*} \mu(\overline{\zeta})\overline{K(z,\zeta)}\ dA(\zeta),\ z \in \mathbb{H}^*,$$

defines an element in $\mathrm{QD}^*(\Gamma)$. Indeed, by the transformation formula,

$$\Phi[\mu](\gamma(z))\gamma'(z)^2 = \int_{\gamma(\mathbb{H}^*)} \mu(\overline{\zeta})\overline{K(\gamma(z),\zeta)} dA(\zeta)\gamma'(z)^2 = \int_{\mathbb{H}^*} \mu(\gamma(\overline{\zeta}))\overline{K(\gamma(z),\gamma(\zeta))}\gamma'(z)^2|\gamma'(\zeta)|^2\ dA(\zeta) =$$
$$= \int_{\mathbb{H}^*} \underbrace{\mu(\gamma(\overline{\zeta}))\frac{\gamma'(\zeta)}{\gamma'(\overline{\zeta})}}_{=\mu(\overline{\zeta})} \frac{\gamma'(\overline{\zeta})}{\gamma'(\zeta)} \underbrace{\overline{K(\gamma(z),\gamma(\zeta))}\gamma'(z)^2\overline{\gamma'(\zeta)^2}}_{=\overline{K(z,\zeta)}} \frac{\gamma'(\zeta)}{\gamma'(\overline{\zeta})}\ dA(\zeta) = \int_{\mathbb{H}^*} \mu(\overline{\zeta})\overline{K(z,\zeta)}\ dA(\zeta) = \Phi[\mu](z),$$

for any $\gamma \in \Gamma$. We will need the following lemma, which motivates the terminology "reproducing kernel".

Lemma 2.25 (The Reproducing Formula)**.** *For any $\phi \in \mathrm{QD}(\Gamma)$ and any $z \in \mathbb{H}$, we have*

$$\phi(z) = \int_{\mathbb{H}} \Im(\zeta)^2\phi(\zeta)\overline{K(z,\zeta)}\ dA(\zeta)$$

and, for any $\phi \in \mathrm{QD}^*(\Gamma)$ and any $z \in \mathbb{H}^*$, we have

$$\phi(z) = \int_{\mathbb{H}^*} \Im(\zeta)^2\phi(\zeta)\overline{K(z,\zeta)}\ dA(\zeta).$$

Proof. First of all, we show that the formula for $\mathrm{QD}^*(\Gamma)$ follows from the one for $\mathrm{QD}(\Gamma)$ by remark 2.11. Indeed, suppose that $\phi(z) = \overline{\psi(\overline{z})} \in \mathrm{QD}^*(\Gamma)$ with $\psi \in \mathrm{QD}(\Gamma)$ and that the formula is proved for $\mathrm{QD}(\Gamma)$. Then we easily see

$$\phi(z) = \overline{\int_{\mathbb{H}} \Im(\zeta)^2\psi(\zeta)\overline{K(\overline{z},\zeta)}\ dA(\zeta)} = \int_{\mathbb{H}^*} \Im(\overline{\zeta})^2\overline{\psi(\overline{\zeta})}K(\overline{z},\overline{\zeta})\ dA(\zeta) = \int_{\mathbb{H}^*} \Im(\zeta)^2\phi(\zeta)\overline{K(z,\zeta)}\ dA(\zeta).$$

Thus, it suffices to show the first reproducing formula. Moreover, let us verify that we only need to prove the formula for $z = i$. Suppose the formula holds for all $\phi \in \mathrm{QD}(\Gamma)$ for $z = i$ and that an arbitrary point $z \in \mathbb{H}$ is given. Take a Möbius transformation γ with $\gamma(z) = i$. Then

$$\phi(\gamma(z)) = \int_{\mathbb{H}} \Im(\zeta)^2 \phi(\zeta)\overline{K(\gamma(z),\zeta)}\ dA(\zeta) = \int_{\mathbb{H}} \underbrace{\Im(\gamma(\zeta))^2}_{=\Im(\zeta)^2\gamma'(\zeta)\gamma'(\overline{\zeta})} \phi(\gamma(\zeta))\overline{K(\gamma(z),\gamma(\zeta))}|\gamma'(\zeta)|^2\ dA(\zeta) =$$

$$= \int_{\mathbb{H}} \Im(\zeta)^2 \underbrace{\phi(\gamma(\zeta))\gamma'(\zeta)^2}_{=\phi(\zeta)} \underbrace{\overline{\gamma'(\zeta)^2 K(\gamma(z),\gamma(\zeta))\gamma'(z)^2}}_{=\overline{K(z,\zeta)}} \frac{1}{\gamma'(z)^2}\ dA(\zeta) = \frac{1}{\gamma'(z)^2}\int_{\mathbb{H}} \Im(\zeta)^2\phi(\zeta)\overline{K(z,\zeta)}\ dA(\zeta)$$

and, hence,

$$\int_{\mathbb{H}} \Im(\zeta)^2\phi(\zeta)\overline{K(z,\zeta)}\ dA(\zeta = \phi(\gamma(z))\gamma'(z)^2 = \phi(z),$$

as desired. It remains to prove the formula for $z = i$. We would like to work in the unit disk Δ instead of $\mathbb{H}$. Let S be the map $z \mapsto i\frac{1+z}{1-z}$, which maps Δ to $\mathbb{H}$. Transforming the integral on the right hand side of the reproducing formula with respect to S with $z = i$ yields (after a lengthy computation similar in spirit to the previous ones)

$$\int_{\mathbb{H}} \Im(\zeta)^2\phi(\zeta)\overline{K(i,\zeta)}\ dA(\zeta) = -\frac{3}{4\pi}\int_{\Delta} \left((1-|\zeta|^2\right)^2 \tilde{\phi}(\zeta)\ dA(\zeta),$$

where $\tilde{\phi}(z) = (\phi \circ S)(z) \cdot S'(z)^2$ is a holomorphic quadratic differential on Δ with respect to the Fuchsian group $S^{-1}\Gamma S$. Thus, the reproducing formula for $\mathbb{H}$ is equivalent to

$$-\frac{3}{4\pi}\int_{\Delta} \left((1-|\zeta|^2\right)^2 \tilde{\phi}(\zeta)\ dA(\zeta) = \phi(i) = \frac{1}{S'(0)^2}\tilde{\phi}(0) = -\frac{1}{4}\tilde{\phi}(0).$$

However, the equation

$$\tilde{\phi}(0) = \frac{3}{\pi}\int_{\Delta} \left((1-|\zeta|^2\right)^2 \tilde{\phi}(\zeta)\ dA(\zeta)$$

is trivial. Indeed, if we write $\tilde{\phi}$ in its power series expansion $\sum_{n\geq 0} a_n z^n$, then the integral on the right becomes

$$\sum_{n\geq 0} a_n \int_{\Delta} \left((1-|\zeta|^2\right)^2 \zeta^n\ dA(\zeta) = \sum_{n\geq 0} a_n \int_0^1 \left(1-r^2\right)^2 r^{n+1} \underbrace{\int_0^{2\pi} e^{in\pi\phi}\ d\phi}_{=0 \text{ for } n\geq 1}\ dr =$$

$$= a_0 2\pi \int_0^1 \left(1-r^2\right)^2 r\ dr = \tilde{\phi}(0)\frac{2\pi}{6}.$$

□

We can now prove the remark about the cotangent bundle of $\mathrm{Teich}(\Gamma)$.

Proposition 2.26. *There exists a splitting of vector spaces*

$$\mathrm{BC}(\Gamma) = \mathrm{HBC}(\Gamma) \oplus N(\Gamma),$$

where $\mathrm{HBC}(\Gamma)$ *is the image of* $\mathrm{QD}^*(\Gamma)$ *under the map* $H[\phi](z) = \Im(z)^2\phi(\overline{z})$ *and* $N(\Gamma)$ *is the kernel of*

$$\Lambda\colon \mathrm{BC}(\Gamma) \to \mathrm{QD}^*(\Gamma)^*,\ \mu \mapsto (\mu,\cdot),\ (\mu,\phi) = \int_F \mu(z)\overline{\phi(\overline{z})}\ dA(z).$$

Proof. First note that, for any element $\phi \in \mathrm{QD}^*(\Gamma)$, the equality $(H[\phi], \phi) = 0$ implies $\phi \equiv 0$ and, hence, $H[\phi] \equiv 0$. Consequently, it follows immediately that $\mathrm{HBC}(\Gamma)$ and $N(\Gamma)$ have zero intersection. Let us show that $\mathrm{HBC}(\Gamma) + N(\Gamma) = \mathrm{BC}(\Gamma)$. Consider the map

$$\Psi \colon \mathrm{BC}(\Gamma) \to \mathrm{BC}(\Gamma),\ \mu \mapsto \Psi[\mu](z) = \Im(z)^2 \Phi[\mu](\overline{z}).$$

Since Φ takes values in $\mathrm{QD}^*(\Gamma)$, $\Psi(\mu) \in \mathrm{HBC}(\Gamma)$ holds for every $\mu \in \mathrm{BC}(\Gamma)$, by definition of $\mathrm{HBC}(\Gamma)$. Thus, writing $\mu \in \mathrm{BC}(\Gamma)$ as $\Psi[\mu] + (\mu - \Psi[\mu])$, it suffices to show that $\mu - \Psi[\mu] \in N(\Gamma)$. Note that, by the reproducing formula, we have $\Phi \circ \Psi = \Phi$. In particular, it follows from the definition of Ψ that $\Psi^2 = \Psi$. Therefore, we always have $\mu - \Psi[\mu] \in \mathrm{kernel}(\Psi)$, and the proof reduces to the claim:

$$N(\Gamma) = \mathrm{kernel}(\Psi) = \mathrm{kernel}(\Phi).$$

As in the proof of the last lemma, we would like to work with the unit disk Δ and its complement $\Delta^* = \mathbb{C} \setminus \overline{\Delta}$ instead of $\mathbb{H}$ and $\mathbb{H}^*$. Let S be the transformation $z \mapsto i\frac{1+z}{1-z}$ and, moreover, set $T(z) = \overline{S(z)}$. Then S maps Δ to $\mathbb{H}$ and Δ^* to $\mathbb{H}^*$ and T maps Δ to $\mathbb{H}^*$ and Δ^* to $\mathbb{H}$. Further, we define

$$\begin{aligned}
\nu(z) &= (\mu \circ S(z)) \cdot \frac{\overline{S'(z)}}{S'(z)},\ z \in \Delta, \\
\tilde{\Phi}(z) &= (\Phi[\mu] \circ S(z)) \cdot S'(z)^2,\ z \in \Delta^*.
\end{aligned}$$

Then ν is a Beltrami coefficient on the unit disk with respect to the Fuchsian group $\Gamma' = S^{-1}\Gamma S$, and $\tilde{\Phi}$ is a holomorphic quadratic differential on Δ^* with respect to Γ'. We transform the integral that defines $\tilde{\Phi}$ with respect to T to obtain a different integral formula:

$$\begin{aligned}
\tilde{\Phi}(z) &= \int_{T(\Delta)} \mu(\overline{\zeta}) \overline{K(S(z), \zeta)} S'(z)^2 \, dA(\zeta) = \int_\Delta \mu(\overline{T(\zeta)}) \overline{K(S(z), T(\zeta))} S'(z)^2 \underbrace{|\det(DT(\zeta))|}_{=|S'(\zeta)|^2} \, dA(\zeta) = \\
&= \int_\Delta \mu(S(\zeta)) \overline{K(S(z), \overline{S(\zeta)})} S'(z)^2 |S'(\zeta)|^2 \, dA(\zeta) = \frac{12}{\pi} \int_\Delta \nu(\zeta) \underbrace{\frac{1}{(S(z) - S(\zeta))^4} S'(z)^2 S'(\zeta)^2}_{=(z-\zeta)^{-4}} \, dA(\zeta) = \\
&= \frac{12}{\pi} \int_\Delta \nu(\zeta) \frac{1}{(z-\zeta)^4} \, dA(\zeta).
\end{aligned}$$

The power series expansion of $\tilde{\Phi}$ around 0 has the form

$$\tilde{\Phi}(z) = \sum_{n \in \mathbb{Z}} a_n z^n,\ a_n = \frac{1}{2\pi i} \int_c \frac{\tilde{\Phi}(z)}{z^{n+1}} \, dz,$$

where c is a closed curve in Δ^* that winds around the origin once. By the residue theorem, we have

$$\frac{1}{2\pi i} \int_c \underbrace{\frac{1}{(z-\zeta)^4} \frac{1}{z^{n+1}}}_{\text{denoted } f(z)} \, dz = \begin{cases} \mathrm{Res}(f, \zeta) + \mathrm{Res}(f, 0) = 0, & \text{if } n \geq 0, \\ \mathrm{Res}(f, \zeta) = 0, & \text{if } -3 \leq n \leq -1, \\ \mathrm{Res}(f, \zeta) = -\frac{1}{6}(n+1)(n+2)(n+3)\zeta^{-(n+4)}, & \text{if } n \leq -4. \end{cases}$$

We can use these to calculate the expansion coefficients. They are 0 for $n \geq -3$ and for $n \leq -4$ we get

$$a_n = \frac{12}{\pi} \int_\Delta \nu(\zeta) \left(\frac{1}{2\pi i} \int_c \frac{1}{(z-\zeta)^4} \frac{1}{z^{n+1}} \, dz \right) dA(\zeta) = -\frac{2}{\pi}(n+1)(n+2)(n+3) \int_\Delta \nu(\zeta) \zeta^{-(n+4)} \, dA(\zeta).$$

Rewriting the sum in positive terms, we conclude that

$$\tilde{\Phi}(z) = -\frac{2}{\pi} \sum_{n \geq 4} (n-1)(n-2)(n-3) z^{-n} \int_{\Delta} \nu(\zeta) \zeta^{n-4} \, dA(\zeta).$$

Thus, we have $\mu \in \text{kernel}(\Phi)$ if and only if $\tilde{\Phi} \equiv 0$ if and only if for every $n \geq 4$:

$$\int_{\Delta} \nu(\zeta) \zeta^{n-4} \, dA(\zeta) = 0.$$

Furthermore, this equality holds for all monomials ζ^{n-4}, $n \geq 4$, if and only if

$$\int_{\Delta} \nu(\zeta) f(\zeta) \, dA(\zeta) = 0$$

holds for all integrable holomorphic functions $f \colon \Delta \to \mathbb{C}$. Recall that, by remark 2.11, $\mu \in N(\Gamma)$ if and only if for every $\phi \in \mathrm{QD}(\Gamma)$ the integral

$$\int_{F} \mu(z) \phi(z) \, dA(z)$$

vanishes, where F is any fundamental domain of Γ in $\mathbb{H}$, which is the case if and only if for every $\phi \in \mathrm{QD}(\Gamma)$ the integral

$$\int_{\mathbb{H}} \mu(z) \phi(z) \, dA(z)$$

vanishes. Transforming this integral with respect to S similar to above, it becomes

$$\int_{\mathbb{H}} \mu(z) \phi(z) \, dA(z) = \int_{\Delta} \nu(\zeta) \phi(S(\zeta)) S'(\zeta)^2 \, dA(\zeta) = \int_{\Delta} \nu(\zeta) \tilde{\phi}(\zeta) \, dA(\zeta),$$

where $\tilde{\phi}$ is the holomorphic quadratic differential $(\phi \circ S) \cdot S'^2$ on Δ with respect to Γ' obtained from ϕ. Clearly, we are finished proving one inclusion:

$$\begin{aligned} \mu \in \text{kernel}(\Phi) \ &\Leftrightarrow\ \forall \text{ integrable holomorphic } f \colon \int_{\Delta} \nu(\zeta) f(\zeta) \, dA(\zeta) = 0 \\ &\Rightarrow\ \forall \tilde{\phi} \in \mathrm{QD}^*(\Gamma', \Delta) \colon \int_{\Delta} \nu(\zeta) \tilde{\phi}(\zeta) \, dA(\zeta) = 0 \ \Leftrightarrow\ \mu \in N(\Gamma). \end{aligned}$$

Now suppose that the last integral vanishes for every holomorphic quadratic differential $\tilde{\phi}$ on Δ with respect to Γ' and that we are given an integrable holomorphic function $f \colon \Delta \to \mathbb{C}$. Consider the *Poincaré series of f for Γ'*

$$\Theta(f)(\zeta) = \sum_{\gamma \in \Gamma'} f(\gamma(\zeta)) \gamma'(\zeta)^2.$$

We claim that $\Theta(f)$ is a holomorphic quadratic differential on Δ with respect to Γ'. That we have $\Theta(f)(z) = (\Theta(f) \circ \gamma(z)) \cdot \gamma'(z)^2$ for all $\gamma \in \Gamma'$ follows from the definition. We need to check that the series defining $\Theta(f)$ converges absolutely and uniformly on compact sets so that $\Theta(f)$ is a well-defined holomorphic function. Once this is shown, the proof is complete because then (for some fixed fundamental domain $F \subset \Delta$ of Γ')

$$\begin{aligned} \int_{\Delta} \nu(\zeta) f(\zeta) \, dA(\zeta) &= \sum_{\gamma \in \Gamma'} \int_{\gamma \cdot F} \nu(\zeta) f(\zeta) \, dA(\zeta) = \sum_{\gamma \in \Gamma'} \int_{F} \nu(\gamma(\zeta)) f(\gamma(\zeta)) |\gamma'(\zeta)|^2 \, dA(\zeta) = \\ &= \sum_{\gamma \in \Gamma'} \int_{F} \nu(\zeta) \frac{\overline{\gamma'(\zeta)}}{\gamma'(\zeta)} f(\gamma(\zeta)) \gamma'(\zeta) \overline{\gamma'(\zeta)} \, dA(\zeta) = \int_{F} \nu(\zeta) \Theta(f)(\zeta) \, dA(\zeta), \end{aligned}$$

and the last integral vanishes by hypothesis. Since, in the very beginning of this chapter, we chose Γ such that $\mathbb{H}/\Gamma$ is a closed Riemann surface, no element in Γ other than the identity has a fixed point in $\mathbb{H}$. Thus, for any given compact set $K \subset \Delta$, we can pick $r > 0$ so small that $\gamma(B(\zeta, r)) \cap B(\zeta, r) = \emptyset$, for every $\zeta \in K$ and every $\gamma \in \Gamma' \setminus \{\mathrm{id}\}$. Since the absolute value of a holomorphic function is subharmonic, we can apply the mean value theorem for subharmonic functions to $|f(\gamma(\zeta))\gamma'(\zeta)^2|$ to obtain the bound

$$|f(\gamma(\zeta))\gamma'(\zeta)^2| \leq \frac{1}{\pi r^2}\int_{B(\zeta,r)} |f(\gamma(w))\gamma'(w)^2|\, dA(w) = \frac{1}{\pi r^2}\int_{\gamma(B(\zeta,r))} |f(w)|\, dA(w).$$

We conclude that

$$|\Theta(f)(\zeta)| \leq \sum_{\gamma\in\Gamma'} \frac{1}{\pi r^2}\int_{\gamma(B(\zeta,r))} |f(w)|\, dA(w) \leq \frac{1}{\pi r^2}\int_{\Delta} |f(w)|\, dA(w) < \infty$$

uniformly for $\zeta \in K$, which shows that $\Theta(f)$ is holomorphic and finishes the proof. □

Remark 2.27. *Note that, by the reproducing formula, we have $\Phi \circ \Psi = \Phi$ and $\Phi \circ H = \mathrm{id}_{\mathrm{QD}^*(\Gamma)}$ as well as $\Psi \circ H = H$.*

The second result we want to prove in this chapter is that the Teichmüller metric is indeed a metric and that it is complete. Recall its definition from chapter 1.1. Given any two points $[(X, \phi)], [(Y, \psi)] \in \mathrm{Teich}(\mathcal{S})$, we let $\mathcal{F}$ denote the set of quasi-conformal maps $X \to Y$ that are isotopic to $\psi \circ \phi^{-1}$ and define

$$\mathrm{d}_{\mathrm{Teich}}([X], [Y]) = \inf_{h\in\mathcal{F}} \log(K(h))/2,$$

where $K(h)$ is the maximal dilatation of h.

Lemma 2.28. $\mathrm{d}_{\mathrm{Teich}}$ *is a metric on* $\mathrm{Teich}(\mathcal{S})$*.*

Proof. Clearly, $\mathrm{d}_{\mathrm{Teich}}$ is well-defined because if we take different representatives of the points, then the set $\mathcal{F}$ remains unchanged. Symmetry and the triangle inequality follow readily from the properties of the dilatation. We only need to prove that $\mathrm{d}_{\mathrm{Teich}}([X], [Y]) = 0$ implies $[X] = [Y]$. If $\mathrm{d}_{\mathrm{Teich}}([(X, \phi)], [(Y, \psi)])$ is zero, then there is a sequence $(f_n)_{n\geq 0} \subset \mathcal{F}$ with $K(f_n) \to 1$ or, equivalently, $\mu(f_n^*) \to 0$, where f_n^* is a lift of f_n. Let γ_n be the Möbius transformation so that $\gamma_n^{-1} \circ f_n^*$ agrees with $w_{\mu(f_n^*)}$ on $\mathbb{H}$. Without loss of generality, we can take the sequence $(f_n)_{n\geq 0}$ such that γ_n is the same transformation γ_0 for all $n \geq 0$. By lemma 2.12, $(\gamma_0^{-1} \circ f_n^*)_{n\geq 0}$ converges locally uniformly to the identity. To shorten notation, set $f = \psi \circ \phi^{-1}$. Since f_n is isotopic to f, we have, for every $\gamma \in \Gamma(X)$, $X = \mathbb{H}/\Gamma(X)$,

$$f^* \circ \gamma \circ (f^*)^{-1} = f_n^* \circ \gamma \circ (f_n^*)^{-1} \longrightarrow \gamma_0 \circ \gamma \circ \gamma_0^{-1} \text{ as } n \to \infty.$$

Thus, if $[f]_*$ and $[\gamma_0]_*$ denote the induced maps between the Fuchsian groups, then $[f]_* = [\gamma_0]_*$. In particular, γ_0 descends to a map $X \to Y$. As in one of the previous proofs, if c is the geodesic between $f(z)$ and $\gamma_0(z)$, $z \in \mathbb{H}$, then $[f]_*(\gamma)(c)$ is the geodesic between $[f]_*(\gamma)\circ f(z) = f\circ\gamma(z)$ and $[f]_*(\gamma)\circ\gamma_0(z) = \gamma_0 \circ \gamma(z)$. Therefore, if we denote the former geodesic by $F_t^*(z) = c(t)$, then F_t^* factors through $\Gamma(X)$,

$$F_t^* \circ \gamma(z) = [f]_*(\gamma) \circ F_t^*(z),$$

and, hence, descends to an isotopy $F_t \colon \mathbb{H}/\Gamma(X) \to \mathbb{H}/\Gamma(Y)$ between $\psi \circ \phi^{-1}$ and the quotient map of γ_0. This shows that $\psi \circ \phi^{-1}$ is isotopic to a conformal map, i.e. $[(X, \phi)] = [(Y, \psi)]$ in $\mathrm{Teich}(\mathcal{S})$. □

Theorem 2.29. $\mathrm{d}_{\mathrm{Teich}}$ *is a complete metric on* $\mathrm{Teich}(\mathcal{S})$*.*

Proof. Let $\{[(X_n, \phi_n)]\}_{n\geq 0} \subset \mathrm{Teich}(\mathcal{S})$ be a Cauchy sequence with respect to $\mathrm{d}_{\mathrm{Teich}}$. Thus, for all $\epsilon > 0$ there exists some $N \geq 0$ such that for all $n, m \geq N$ there is some quasi-conformal map $\phi_{n,m}$ isotopic to $\phi_m \circ \phi_n^{-1}$ with $\log(K(\phi_{n,m})) < \epsilon$. The condition that $\log(K(\phi_{n,m}))$ becomes small is equivalent to the condition that the Beltrami coefficient of $\phi_{n,m}$ becomes small in the supremums norm. Hence, after passing to a subsequence $(n_j)_{j\geq 1}$ if necessary, for all $j \geq 1$, there is a quasi-conformal map $f_j = \phi_{n_j,n_{j+1}}$ isotopic to $\phi_{n_{j+1}} \circ \phi_{n_j}^{-1}$ with a Beltrami coefficient $\mu(f_j)$ that is bounded by 2^{-j}. Define a new map isotopic to ϕ_{n_j} by

$$g_j = f_{j-1} \circ f_{j-2} \circ \cdots \circ f_2 \circ f_1 \circ \phi_{n_1} \colon \mathcal{S} \to X_{n_j}.$$

The dilatation of g_j is bounded by

$$K(g_j) \leq K(\phi_{n_1}) \prod_{k=1}^{j-1} K(f_k) = K(\phi_{n_1}) \prod_{k=1}^{j-1} \frac{1+\mu(f_k)}{1-\mu(f_k)} \leq K(\phi_{n_1}) \prod_{k=1}^{j-1} \frac{1+2^{-k}}{1-2^{-k}} \leq 2K(\phi_{n_1}) \prod_{k=1}^{j-1} (1+2^{-k}),$$

which is uniformly bounded in j. Therefore, we can take the supremum of all $K(g_j)$, $j \geq 1$, which we denote by K. Consequently, the Beltrami coefficients $\mu_j \in \mathrm{BC}(\Gamma)_1$, induced from g_j by taking the Beltrami coefficient of a lift g_j^*, are uniformly bounded by $K_0 = \frac{K-1}{K+1}$. Moreover, by lemma 2.1, we see that $(\mu_j)_{j\geq 1}$ is a Cauchy sequence in $\mathrm{BC}(\Gamma)_1$:

$$||\mu_{j+1} - \mu_j||_\infty \leq \left|\left|\frac{\mu_{j+1} - \mu_j}{1 - \mu_{j+1}\overline{\mu_j}}\right|\right|_\infty = ||\mu(g_{j+1} \circ g_j^{-1})||_\infty = ||\mu(f_j)||_\infty < 2^{-j}.$$

Let μ be the limit point of $(\mu_j)_{j\geq 1}$ in $\mathrm{BC}(\Gamma)_1$, let $f^* \in \mathrm{QC}(\Gamma)$ be the solution of μ obtained from corollary 2.3, and let f be the quotient map determining a point in $\mathrm{Teich}(\mathcal{S})$. Using lemma 2.1 again, we see that

$$||\mu(f \circ g_j^{-1})||_\infty = \left|\left|\frac{\mu - \mu_j}{1 - \mu\overline{\mu_j}}\right|\right|_\infty \leq \frac{1}{1-K_0^2} ||\mu - \mu_j||_\infty \longrightarrow 0 \text{ as } j \to \infty.$$

Thus, the sequence $(\phi_{n_j})_{j\geq 1}$ converges to the quotient map f:

$$\exp(2\mathrm{d}_{\mathrm{Teich}}(f, \phi_{n_j})) \leq (K(f \circ g_j^{-1})) = \frac{1 + ||\mu(f \circ g_j^{-1})||_\infty}{1 - ||\mu(f \circ g_j^{-1})||_\infty} \longrightarrow 1,$$

where we used that g_j is isotopic to ϕ_{n_j}. □

To finish the proof of theorem 1.3, we need an argument to justify that $\mathrm{d}_{\mathrm{Teich}}$ induces the topology of any Fenchel-Nielsen coordinates. As observed in the previous proof, a small perturbation in the Teichmüller metric corresponds to a change-of-marking map with small Beltrami coefficient. Since the Beltrami coefficient measures the distortion between the complex structures, a small perturbation in $\mathrm{d}_{\mathrm{Teich}}$ corresponds to a small perturbation of the complex structure. However, the Fenchel-Nielsen coordinates are constructed to measure the same quantity. Thus, we end up with a small perturbation in FN-coordinates. The argument works the other way around, as well. Thus, the topology is the same in both cases. In particular, it does not depend on the choice of Fenchel-Nielsen coordinates.

3 An Application of the Weil-Petersson Metric

3.1 The Pants Graph

Let P and P' be two pants decompositions of $\mathcal{S}_g$. Suppose that we can find geodesics $\gamma \in P$ and $\delta \in P'$ such that $P \setminus \{\gamma\} = P' \setminus \{\delta\}$. Then the other $3g-4$ geodesics in P and P', respectively, are identical. Hence, if cut along the other $3g-4$ geodesics, then a once punctured torus $\mathcal{S}_{1,1}$ or a four times punctured sphere $\mathcal{S}_{0,4}$ remains and γ and δ are strictly contained in this remaining component. We say that P' can be obtained from P by an *elementary move* if we can find geodesics $\gamma \in P$ and $\delta \in P'$ as above and such that the following holds: if the remaining component is a once punctured torus, then γ and δ intersect exactly once; if the remaining component is a four times punctured sphere, then γ and δ intersect exactly twice.

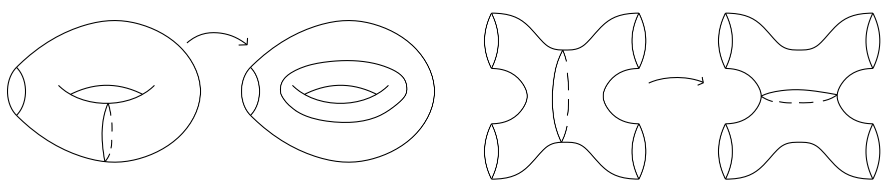

Figure 5: Elementary moves of a pants decomposition.

Let us briefly discuss which two pants decompositions are related by an elementary move. Suppose P and P' are related and $P \setminus \{\gamma\} = P' \setminus \{\delta\}$. What are minimally intersecting geodesics in $\mathcal{S}_{1,1}$ and $\mathcal{S}_{0,4}$? Let us consider the once punctured torus first. We want to understand which homotopy type γ and δ can have. The homotopy classes of γ and δ in $\mathcal{S}_{1,1}$ are actually the same as their homotopy classes in the torus. Indeed, if c is a curve that is homotopic to γ in the torus, then c and γ bound two annuli, by theorem A.2. Here, we used that γ is non-peripheral. Using the annulus that does not contain the puncture, we see that c and γ are also homotopic in the punctured torus. Knowing that it does not matter whether we consider the homotopy classes in $\mathcal{S}_{1,1}$ or in $\mathcal{S}_{1,0}$, we can conclude that the intersection number of γ and δ in the torus is the same as their intersection number in the punctured torus, i.e. one. Now suppose that the homotopy classes of γ and δ in $\mathcal{S}_{1,0}$ are represented by $(p,q) \in \mathbb{Z} \times \mathbb{Z} \simeq \pi_1(\mathcal{S}_{1,0}, *)$ and (p', q'), respectively. Due to symmetry, we may assume without loss of generality that $p \neq 0$, and we distinguish the two cases $q = 0$ and $q \neq 0$. If $q = 0$, then δ must be of the form $(p', 1)$ since it intersects γ exactly once. If $q \neq 0$, then consider the integer matrix

$$A = \begin{bmatrix} a & b \\ -q & p \end{bmatrix},$$

where $a, b \in \mathbb{Z}$ are chosen such that $ap + bq = 1$. As an integer matrix with determinant one, A represents a diffeomorphism of the torus (viewed as the quotient $\mathbb{R}^2/\mathbb{Z}^2$). Note that it maps the homotopy class (p,q) to the homotopy class $(1,0)$. Since diffeomorphisms preserve the intersection number, $A(\gamma)$ and $A(\delta)$ intersect exactly once. As $A(\gamma) \simeq (1,0)$, $A(\delta)$ must be of the form $(m,1)$, for some non-negative integer m. Hence, δ is of the form

$$\begin{bmatrix} p' \\ q' \end{bmatrix} = A^{-1} \cdot \begin{bmatrix} m \\ 1 \end{bmatrix} = \begin{bmatrix} p & -b \\ q & a \end{bmatrix} \cdot \begin{bmatrix} m \\ 1 \end{bmatrix} = \begin{bmatrix} mp - b \\ mq + a \end{bmatrix},$$

where m is some non-negative integer. Now consider the case of a four times punctured sphere. In fact, this case is not different than the first one because the homotopy classes of essential, non-peripheral, simple, closed curves in $\mathcal{S}_{0,4}$ are in bijection with the homotopy classes of essential, simple, closed curves in the torus $\mathcal{S}_{1,0}$ (see [6, p, 55]). Under this bijection, the class $(1,0) \in \pi_1(\mathcal{S}_{1,0})$ corresponds to the loop that winds once around one great circle in $\mathcal{S}_{0,4}$, and $(0,1) \in \pi_1(\mathcal{S}_{1,0})$ corresponds to the loop winding once around the other great circle in $\mathcal{S}_{0,4}$. Thus, the bijection pairwise preserves the intersection number up to factor of 2. This factor is corrected by the fact that minimally intersecting geodesics in $\mathcal{S}_{0,4}$ intersect exactly twice. Hence, if $\gamma \in P$ is represented by $(p,q) \in \pi_1(\mathcal{S}_{1,0})$, then $\delta \in P'$ must be represented by $(p',1)$ if $q=0$ or by $(mp-b, mq+a)$ if $q \neq 0$, where $a, b \in \mathbb{Z}$ solve the equation $ap+bq=1$ and m is any non-negative integer. Having understood what it means for two pants decompositions to be related by an elementary move, we can proceed to the main theorem of this chapter. We can define the *pants graph* $\mathcal{P}_g$ by taking the different pants decompositions of $\mathcal{S}_g$ as vertices and link two vertices by an edge if they are related by one elementary move. We give each edge distance one, which makes the set of vertices $\mathcal{P}_g V$ a metric space. We are interested in the following result.

Theorem 3.1. *The vertex space $\mathcal{P}_g V$ of the pants graph is quasi-isometric to the Teichmüller space equipped with the Weil-Petersson metric.*

Let us recall the definition of a quasi-isometry. A (k_1,k_2)-quasi-isometric embedding, $k_1 \geq 1$ and $k_2 \geq 0$, between two metric spaces (X, d_X) and (Y, d_Y) is a map $f\colon X \to Y$ satisfying

$$\frac{d_X(x,x')}{k_1} - k_2 \leq d_Y(f(x), f(x')) \leq k_1 d_X(x,x') + k_2,$$

for all $x, x' \in X$. Two metric spaces (X, d_X) and (Y, d_Y) are (k_1,k_2)-quasi-isometric if there exist (k_1,k_2)-quasi-isometric embeddings $f\colon X \to Y$ and $g\colon Y \to X$ such that the compositions $f \circ g$ and $g \circ f$ are a uniformly bounded distance away from the identity map, i.e. there exists $L > 0$ such that for all $x \in X$ we have $d_X(g \circ f(x), x) \leq L$ and similarly for $f \circ g$. Naturally, X and Y are quasi-isometric if they are (k_1,k_2)-quasi-isometric for some $k_1 \geq 1$ and $k_2 \geq 0$. Let us stress the following important observation.

Remark 3.2. *We call a map as above a quasi-isometric embedding in order to be consistent with the main reference for this chapter, [4]. Beware that we actually did not require a quasi-isometric embedding to be injective, and, thus, it is not an embedding in the usual sense. In fact, it does not even need to be continuous. Counterexamples are constructed easily, for instance, consider the map $\mathbb{R} \to \mathbb{R}$, which is the identity on $\mathbb{R} \setminus \{0\}$, but maps 0 to 1. This map is a (k_1,k_2)-quasi-isometric embedding, for any $k_1 \geq 1$ and any $k_2 \geq 1$. In a way, the parameter k_2 measures the obstruction to being continuous.*

Since we do not require injectivity or continuity, nor do we need inverse maps when talking about quasi-isometric spaces, it suffices to find a single quasi-isometry that also happens to have a sort of dense image, as the next lemma shows.

Lemma 3.3. *If $f\colon X \to Y$ is a (k_1,k_2)-quasi-isometric embedding such that its image is D-dense (i.e. for every $y \in Y$ there exists $x \in X$ with $d_Y(y, f(x)) \leq D$), then (X, d_X) and (Y, d_Y) are $(k_1, k_1(2D+k_2))$-quasi-isometric.*

Proof. We need to construct a $(k_1, k_1(2D+k_2))$-quasi-isometric embedding $g\colon Y \to X$. We go with the obvious choice: by hypothesis, for all $y \in Y$ there exists a point $x \in X$, which we call $g(y)$, with $d_Y(y, f(x)) \leq D$. Clearly, this definition of g is not unique and g is not injective, but this does not bother us. All we need to do is to compute that g satisfies the required inequalities. These follow immediately from the definition of g, the inequalities for f, and the triangle inequality,

$$d_Y(y,y') \leq \underbrace{d_Y(y, f(g(y)))}_{\leq D} + \underbrace{d_Y(f(g(y)), f(g(y')))}_{\leq k_1 d_X(g(y), g(y')) + k_2} + \underbrace{d_Y(f(g(y')), y')}_{\leq D} \leq k_1 d_X(g(y), g(y')) + (2D + k_2).$$

Similarly, we can get the second inequality for g, as well,

$$\frac{d_X(g(y), g(y'))}{k_1} - k_2 \leq d_Y(f(g(y)), f(g(y'))) \leq$$
$$\leq \underbrace{d_Y(f(g(y)), y)}_{\leq D} + d_Y(y,y') + \underbrace{d_Y(y', f(g(y')))}_{\leq D} \leq d_Y(y,y') + 2D.$$

□

Before stating which map in the main result of this chapter is a quasi-isometry, we introduce so-called sub level sets. Given a pants decomposition P of $\mathcal{S}_g$ and a real number $L > 0$, we define the *sub level set of P and L* to be

$$V_L(P) = \left\{ [X] \in \operatorname{Teich}(\mathcal{S}_g) \;\middle|\; \max_{\gamma \in P} L_{[X]}(\gamma) < L \right\}.$$

Recall the definition of Bers' constant B_g and that it only depends on the genus g of $\mathcal{S}_g$. We denote the sub level set of P and B_g simply by $V(P) = V_{\mathrm{B}_g}(P)$. Bers' theorem 1.15 tells us that $\operatorname{Teich}(\mathcal{S}_g)$ is the union of all $V(P)$, where P ranges over all possible pants decompositions of $\mathcal{S}_g$.

Remark 3.4. *To be precise, this might only be true if we use a non-strict inequality in the definition of $V(P)$. However, we want the sub level sets to be open. To fix this, we can simply take Bers' constant to be slightly larger, for example, we can take $\mathrm{B}_g + \epsilon$ to be the new Bers' constant, where ϵ is arbitrarily small.*

Using the properties of the Weil-Petersson metric, we can write down two consequences for the sub level sets.

Lemma 3.5. *Every sub level set $V_L(P)$ is geodesically convex with respect to d_{WP}.*

Proof. Take $[X], [Y] \in V_L(P)$ and let $\{[X_t]\}_{t \in [0,T]} \subset \operatorname{Teich}(\mathcal{S}_g)$ be the WP-geodesic between them. We want to show that $[X_t] \in V_L(P)$, for all $t \in [0, T]$. By proposition 2.23, the length function $t \mapsto L_{[X_t]}(\gamma)$ is convex, for any $\gamma \in P$. Thus, writing $t = \alpha T$, we can compute

$$L_{[X_t]}(\gamma) = L_{[X_{\alpha T}]}(\gamma) \le (1-\alpha) L_{[X_o]}(\gamma) + \alpha L_{[X_T]}(\gamma) \le \max\left\{L_{[X]}(\gamma), L_{[Y]}(\gamma)\right\} < L,$$

for any $\gamma \in P$. Therefore, $[X_t] \in V_L(P)$ for all $t \in [0, T]$, which proves the statement. □

Lemma 3.6. *The diameter of $V_L(P)$ with respect to d_{WP} is bounded by a constant that depends only on L and g. In particular, the diameter of $V(P)$ is bounded by a constant that depends only on g.*

Proof. Given any pants decomposition $P = \{\gamma_1, \ldots, \gamma_{3g-3}\}$ of $\mathcal{S}_g$, let $[Z] \in \overline{\operatorname{Teich}(\mathcal{S}_g)}$ be the equivalence class of a hyperbolic surface with nodes at every $\gamma_1, \ldots, \gamma_{3g-3}$. By proposition 2.24, there is a neighborhood $U \subset \overline{\operatorname{Teich}(\mathcal{S}_g)}$ around $[Z]$ in which the Weil-Petersson distance takes the form

$$\mathrm{d}_{\overline{\mathrm{WP}}}([X], [Z]) = \sqrt{2\pi L([X])} + \mathcal{O}(L([X])^2),$$

where $L([X]) = L_{[X]}(\gamma_1) + \cdots + L_{[X]}(\gamma_{3g-3})$. Take any two points $[X], [Y] \in V(P)$. By definition of the topology of $\overline{\operatorname{Teich}(\mathcal{S}_g)}$, there are points $[X'], [Y']$ in U that have the twist coordinates

$$\theta_j([X']) = \frac{2\pi}{\epsilon}\theta_j([X]),\ \theta_j([Y']) = \frac{2\pi}{\epsilon}\theta_j([Y]),\ 1 \le j \le 3g-3,$$

respectively, and have length coordinates ϵ in all entries, where ϵ is sufficiently small. Let $\gamma\colon [0,1] \to \operatorname{Teich}(\mathcal{S}_g)$ be the curve from $[X']$ to $[X]$ given by

$$\gamma(t) = \Big(tL_{[X]}(\gamma_1) + (1-t)\epsilon, \ldots, tL_{[X]}(\gamma_{3g-3}) + (1-t)\epsilon,$$
$$\frac{2\pi}{tL_{[X]}(\gamma_1) + (1-t)\epsilon}\theta_1([X]), \ldots, \frac{2\pi}{tL_{[X]}(\gamma_{3g-3}) + (1-t)\epsilon}\theta_{3g-3}([X])\Big).$$

By definition, we have

$$\mathrm{d}_{\mathrm{WP}}([X'],[X]) \leq \int_0^1 (\mathrm{g}_{\mathrm{WP}})_{\gamma(t)}(\gamma'(t),\gamma'(t))\ dt = \int_0^1 (\omega_{\mathrm{WP}})_{\gamma(t)}(\gamma'(t), i\gamma'(t))\ dt.$$

By theorem 2.19, ω_{WP} takes the form

$$\omega_{\mathrm{WP}} = \sum_{1\leq j\leq 3g-3} d\tau_j \wedge dl_j,$$

where l_j and $\tau_j = \frac{l_j}{2\pi}\theta_j$, $1 \leq j \leq 3g-3$, denote the adjusted Fenchel-Nielsen coordinate chart of $\mathrm{Teich}(\mathcal{S}_g)$. Using $\tau_j(\gamma(t)) = \theta_j([X])$ and, hence,

$$\gamma'(t) = \sum_{1\leq j\leq 3g-3} (L_{[X]}(\gamma_j) - \epsilon)\left.\frac{\partial}{\partial l_j}\right|_{\gamma(t)},$$
$$i\gamma'(t) = \sum_{1\leq j\leq 3g-3} -(L_{[X]}(\gamma_j) - \epsilon)\left.\frac{\partial}{\partial \tau_j}\right|_{\gamma(t)},$$

which follows from the discussion proceeding Wolpert's formula, we can estimate

$$\begin{aligned}(\omega_{\mathrm{WP}})_{\gamma(t)}(\gamma'(t), i\gamma'(t)) &= \sum_{1\leq j\leq 3g-3}\left(\underbrace{d\tau_j(\gamma'(t))}_{=0} dl_j(i\gamma'(t)) - \overbrace{d\tau_j(i\gamma'(t))}^{=-(L_{[X]}(\gamma_j)-\epsilon)}\ \underbrace{dl_j(\gamma'(t))}_{=(L_{[X]}(\gamma_j)-\epsilon)}\right)\\ &= \sum_{1\leq j\leq 3g-3}(\underbrace{L_{[X]}(\gamma_j)}_{<L} - \epsilon)^2 \leq (3g-3)L^2.\end{aligned}$$

By the same argument, we also have

$$\mathrm{d}_{\mathrm{WP}}([Y'],[Y]) \leq (3g-3)L^2.$$

Using the expansion of $\mathrm{d}_{\overline{\mathrm{WP}}}$ in U, we conclude with the triangle inequality as follows. If D is a fixed bound for $\sqrt{2\pi(3g-3)L} + \mathcal{O}(((3g-3)L)^2)$, then

$$\begin{aligned}\mathrm{d}_{\mathrm{WP}}([X],[Y]) \leq &\mathrm{d}_{\mathrm{WP}}([X],[X']) + \mathrm{d}_{\mathrm{WP}}([X'],[Z]) + \mathrm{d}_{\mathrm{WP}}([Z],[Y']) + \mathrm{d}_{\mathrm{WP}}([Y'],[Y]) \leq\\ \leq &(3g-3)L^2 + D + D + (3g-3)L^2.\end{aligned}$$

This finishes the proof. □

3.2 The Proof of the Quasi-Isometry

Now we will get started on the main theorem. We want to show that $\mathcal{P}_gV$ is quasi-isometric to $(\mathrm{Teich}(\mathcal{S}_g), \mathrm{d}_{\mathrm{WP}})$. It is not yet established what the quasi-isometry will look like. In fact, it will be in a sense canonical, as we will prove with the following stronger statement.

Theorem 3.7. *Let $\mathcal{Q}\colon \mathcal{P}_gV \to \mathrm{Teich}(\mathcal{S}_g)$ be any injective map that sends a pants decomposition $P \in \mathcal{P}_gV$ to some element in $V(P)$. Then $\mathcal{Q}$ is a (k_1, k_2)-quasi-isometric embedding with D-dense image, for some $k_1 \geq 1$, $2 \geq k_2 \geq 0$, and $D \geq 0$.*

Indeed, lemma 3.3 asserts that this theorem implies theorem 3.1. Since the sub level sets form an exhaustion of $\mathrm{Teich}(\mathcal{S}_g)$, it is an immediate consequence of lemma 3.6 that the map $\mathcal{Q}$ in this theorem has D-dense image for some constant $D > 0$. Therefore, the proof of theorem 3.7 boils down to the next two lemmata.

Lemma 3.8. *The map $\mathcal{Q}$ from theorem 3.7 is $2D$-Lipschitz, where D is the density parameter of the map.*

Lemma 3.9. *For the map $\mathcal{Q}$ from theorem 3.7, there exist $k_1 \geq 1$ and $2 \geq k_2 \geq 0$ such that for all $P, P' \in \mathcal{P}_g V$*

$$\frac{\mathrm{d}_{\mathcal{P}_g}(P, P')}{k_1} - k_2 \leq \mathrm{d}_{\mathrm{WP}}(\mathcal{Q}(P), \mathcal{Q}(P')).$$

We begin by proving the former of these two lemmata. To do so, we first establish a general result about Bers' constant being increasing in some sense.

Lemma 3.10. *Collapse $k \leq 3g-3$ disjoint geodesics in $\mathcal{S}_g$ to nodes and consider one piece of the resulting noded hyperbolic surface. Denote this piece by $\mathcal{S}$. If B is the Bers' constant for $\mathcal{S}$, then $B \leq \mathrm{B}_g$.*

Proof. Extend the geodesics that we collapsed to a pants decomposition P and use this for Fenchel-Nielsen coordinates. Consider the following element $[X]$ of $\overline{\mathrm{Teich}(\mathcal{S}_g)}$: the length coordinates at the geodesics that we collapsed are zero; the length coordinates at the geodesics in P that are not contained in $\mathcal{S}$ are zero; the length and twist coordinates in the remaining entries are fixed non-zero, but arbitrary. The restriction to the latter entries represents a fixed, but arbitrary element $[X']$ of the Teichmüller space of $\mathcal{S}$, see remark 1.26. Let P' denote the geodesics in P with non-zero length entries in the description above, i.e. the geodesics that are strictly contained in $\mathcal{S}$. We need to find a pants decomposition Q' of $\mathcal{S}$ such that $L_{[X']}(\gamma) \leq \mathrm{B}_g$ for all $\gamma \in Q'$. Let ϵ be so small that the collar of a geodesic of length less than ϵ has width at least B_g. Take $[Y] \in \mathrm{Teich}(\mathcal{S}_g)$ ϵ-close to $[X]$. Since the sub level sets cover $\mathrm{Teich}(\mathcal{S}_g)$, there is some pants decomposition Q of $\mathcal{S}_g$ with $[Y] \in V(Q)$. By choice of ϵ, no geodesic in Q can cross one of the collars of the geodesics in $P \setminus P'$. Consequently, the geodesics in Q that intersect $\mathcal{S}$ are, in fact, strictly contained in $\mathcal{S}$. Denote the set of these geodesics by Q'. Then Q' is a pants decomposition of $\mathcal{S}$. Since $[Y] \in V(Q)$, the $L_{[Y]}$-length of the geodesics in Q' is bounded by B_g. As $[Y]$ was ϵ-close to $[X]$, the $L_{[X]}$-length and, hence, the $L_{[X']}$-length of the geodesics in Q' is bounded by $\mathrm{B}_g + \epsilon$. ϵ was arbitrarily small, and we can conclude. $\square$

Now we can prove that the map $\mathcal{Q}$ is Lipschitz.

Proof of lemma 3.8. Suppose P and P' differ by one elementary move. By the triangle inequality, it suffices to show that $\mathrm{d}_{\mathrm{WP}}(\mathcal{Q}(P), \mathcal{Q}(P')) \leq 2D$. Let $\gamma \in P$ and $\delta \in P'$ be the geodesics with $P \setminus \{\gamma\} = P' \setminus \{\delta\}$, and let $\mathcal{S} \subset \mathcal{S}_g$ denote the component in which the elementary move takes place. We distinguish the two cases of $\mathcal{S}$ being a once punctured torus and a four times punctured sphere. Assume we are in the first case. Consider the disk model of the hyperbolic space, and let Z' be an ideal square with order four rotational symmetry about the origin. Let α' and β' denote the geodesics that pass the origin, cross Z' in the middle of two opposite sides, and are perpendicular to another, see figure 6 below. By identifying opposite sides of the square, Z' induces a once punctured torus Z. We denote by α and β the geodesics in Z with lifts α' and β', respectively. Since α' and β' are the shortest geodesics in Z' by construction, we have $L_Z(\alpha) = L_Z(\beta) < B$, where B denotes the Bers' constant for $\mathcal{S}$. By the last lemma, $B \leq \mathrm{B}_g$. Fix P for Fenchel-Nielsen coordinates on $\mathcal{S}_g$. If we let $\phi_0 \colon \mathcal{S} \to Z$ be a homeomorphism that sends γ to α and δ to β, then (Z, ϕ_0) determines an element of the Teichmüller space of $\mathcal{S}$. Next, let $[(X, \phi)]$ be a point in the augmented Teichmüller space of $\mathcal{S}_g$ given by a hyperbolic surface with nodes at $P \setminus \{\gamma\}$ and a homeomorphism $\phi \colon \mathcal{S}_g \to X$ that is ϕ_0 when restricted to $\mathcal{S}$. If $[Y] \in \mathrm{Teich}(\mathcal{S}_g)$ is close to $[X] \in \overline{\mathrm{Teich}(\mathcal{S}_g)}$, then the $L_{[Y]}$-lengths of the geodesics in $P \setminus \{\gamma\}$ are close to zero. Moreover, $L_{[Y]}(\gamma)$ and $L_{[Y]}(\delta)$ are close to $L_Z(\alpha) = L_Z(\beta)$, which is bounded by B and, hence, also by B_g. In particular, $[Y]$ lies in the intersection $V(P) \cap V(P')$, so the latter is non-empty. If D denotes the bound from lemma 3.6, then

$$\mathrm{d}_{\mathrm{WP}}(\mathcal{Q}(P), \mathcal{Q}(P')) \leq \mathrm{diameter}_{\mathrm{d}_{\mathrm{WP}}}(V(P) \cup V(P')) \leq 2D,$$

which finishes the case of a once punctured torus. Now suppose $\mathcal{S}$ is a four times punctured sphere. This case is treated almost exactly as the last one. Let Z' be an ideal hexagon with order six rotational symmetry about the origin. Similarly to before, let α' and β' denote geodesics that pass the origin and cross Z' in the middle of two opposite sides, see figure 6. Using the notation from the figure, we can identify the sides 1 and 4, 2 and 3, as well as 5 and 6 to obtain a four times punctured sphere Z. As before, α and β are the geodesics in Z with lifts α' and β', and, by choice of the latter, we have $L_Z(\alpha) = L_Z(\beta) < B = \mathrm{Bers}(\mathcal{S}) \leq \mathrm{B}_g$. The remaining steps can be taken word for word from the first case. □

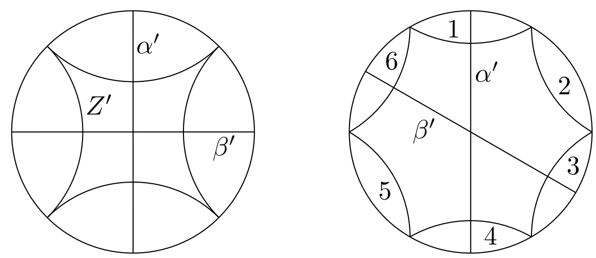

Figure 6: Ideal square and hexagon and shortest geodesics.

We break the proof of the second lemma 3.9 further down into even more lemmata. The first tells us that pants decompositions with overlapping sub level sets have a uniformly bounded distance in $\mathcal{P}_g V$.

Lemma 3.11. *Given $L > \mathrm{B}_g$, there exists $b > 0$ such that for every $P, P' \in \mathcal{P}_g V$ with $V_L(P) \cap V_L(P') \neq \emptyset$ their distance in $\mathcal{P}_g V$ is bounded by b.*

Proof. Take $[X] \in V_L(P) \cap V_L(P')$. By definition of the sub level sets, we have $L_{[X]}(\gamma) < L$, for every $\gamma \in P \cup P'$. By the collar lemma, every $\alpha \in P$ has a collar neighborhood that is disjoint to the collar neighborhoods of the other geodesics in P. If some $\beta \in P'$ intersects $\alpha \in P$ transversely, then β also travels through the collar of α and the section of β that lies inside this collar does not intersect α a second time, nor any other geodesic in P. Since $L_{[X]}(\alpha)$ is bounded from above by L, the width of every collar is bounded from below by some constant $C(L)$. As β also has uniformly bounded length at most L, it can only intersect at most $\lceil L/C(L) \rceil$ geodesics (with repetition) from P transversely. If β intersects a geodesic $\alpha \in P$ non-transversely, then $\beta = \alpha$ by theorem A.2, and we have $i(\alpha, \beta) = 0$, by definition. Because of this, in the remainder of the proof we will implicitly mean transverse intersections whenever we talk about intersections. To shorten notation, denote $C = (3g-3)\lceil L/C(L) \rceil$. We just found a uniform bound

$$i(P, P') = \sum_{\alpha \in P, \beta \in P'} i(\alpha, \beta) \leq \sum_{\beta \in P'} \lceil L/C(L) \rceil = (3g-3)\lceil L/C(L) \rceil = C,$$

where C depends only on the genus g of $\mathcal{S}_g$ and on L. Suppose T is a Dehn twist along some geodesic $\gamma \in P$. If γ' intersects γ k times, then $T \cdot \gamma'$ still intersects γ exactly k times. Thus, the function $i(P, \cdot) \colon \mathcal{P}_g V \to \mathbb{N}_0$ is invariant under composition with T. In particular, if $\mathrm{Tw}(P)$ denotes the subgroup of $\mathrm{MCG}(\mathcal{S}_g)$ generated by Dehn twists along elements in P, then $i(P, \cdot)$ descends to a function on the quotient,

$$i_P \colon \mathcal{P}_g V / \mathrm{Tw}(P) \to \mathbb{N}_0, \ [Q] \mapsto i_P([Q]) = i(P, Q).$$

We claim that the set $Q(P) = \big\{[Q] \in \mathcal{P}_g V/\mathrm{Tw}(P) \;\big|\; i_P([Q]) \leq C\big\}$ is finite. Indeed, if we consider the pairs of pants induced by P, then $i(P, Q) \leq C$ means that the geodesics in Q cross the boundary components of the pairs of pants at most C times. Realizing these C crossings of geodesics in Q with geodesics in P is a finite combinatorial problem and, hence, there are only finitely many possibilities of how to realize these C crossings. The only way to get other pants decompositions with $i(P,Q) \leq C$ is by altering Q without changing the combinatorial data of the crossings, i.e. by altering Q within each pair of pants induced by P. However, this can only be realized by Dehn twists along the boundary components of the pairs of pants, i.e. along P. This shows that the set $Q(P)$ is finite. Next, suppose that P_1 is a pants decomposition that differs from P_2 by Dehn twists $T_1, \dots, T_k$ along geodesics in P. The distance function $\mathrm{d}_{\mathcal{P}_g}$ is invariant under $\mathrm{Hom}^+(\mathcal{S}_g)$ because $\mathrm{Hom}^+(\mathcal{S}_g)$ preserves the geometric intersection number,

$$\mathrm{d}_{\mathcal{P}_g}(P,P') = 1 \iff \mathrm{d}_{\mathcal{P}_g}(f(P), f(P')) = 1, \quad \forall P, P' \in \mathcal{P}_g V \text{ and } \forall f \in \mathrm{Hom}^+(\mathcal{S}_g).$$

Therefore, we have that

$$\mathrm{d}_{\mathcal{P}_g}(P, P_1) = \mathrm{d}_{\mathcal{P}_g}(T_1 \circ \dots \circ T_k(P), T_1 \circ \dots \circ T_k(P_1)) = \mathrm{d}_{\mathcal{P}_g}(P, P_2).$$

This shows that the distance function on $\mathcal{P}_g V$ also descends to the quotient,

$$\mathrm{d}_{\mathcal{P}_g}(P, \cdot)\colon \mathcal{P}_g V/\mathrm{Tw}(P) \to \mathbb{N}_0, \ [Q] \mapsto \mathrm{d}_{\mathcal{P}_g}(P, [Q]) = \mathrm{d}_{\mathcal{P}_g}(P, Q).$$

If $[P']$ denotes the equivalence class of P' in $\mathcal{P}_g V/\mathrm{Tw}(P)$, then we conclude that

$$\mathrm{d}_{\mathcal{P}_g}(P, P') = \mathrm{d}_{\mathcal{P}_g}(P, [P']) \leq \max_{[Q] \in Q(P)} \mathrm{d}_{\mathcal{P}_g}(P, [Q]) = b_P,$$

where the last expression depends on g, L, and P. Lastly, we will show that b_P can be uniformly bounded in P. This follows readily from the fact that there are only finitely many different pants decompositions $P_1, \dots, P_n$ up to orientation-preserving homeomorphisms and the latter preserve geometric intersection of geodesics. Let us make this more precise. To simplify notation, let $[Q]_P$ denote the equivalence class of $Q \in \mathcal{P}_g V$ modulo $\mathrm{Tw}(P)$, $P \in \mathcal{P}_g V$. Note that, for any $f \in \mathrm{Hom}^+(\mathcal{S}_g)$, we have $f([Q]_P) = [f(Q)]_{f(P)}$. Since the intersection number is preserved by $\mathrm{Hom}^+(\mathcal{S}_g)$, we have

$$i_P([f^{-1}(Q)]_P) = i(P, f^{-1}(Q)) = i(f(P), Q) = i_{f(P)}([Q]_{f(P)}).$$

Thus, we have equalities of sets:

$$Q(f(P)) = \big\{[Q]_{f(P)} \bigm| i_{f(P)}([Q]_{f(P)}) \leq C\big\} = \big\{f([f^{-1}(Q)]_P) \bigm| i_P([f^{-1}(Q)]_P) \leq C\big\} = f(Q(P)).$$

Now given $P \in \mathcal{P}_g V$, take $1 \leq k \leq n$ and $f \in \mathrm{Hom}^+(\mathcal{S}_g)$ such that $f(P_k) = P$. Combining the previous results yields

$$b_P = \max_{[Q]_P \in Q(P)} \mathrm{d}_{\mathcal{P}_g}(P, [Q]_P) = \max_{[Q]_{f(P_k)} \in f(Q(P_k))} \mathrm{d}_{\mathcal{P}_g}(f(P_k), [Q]_{f(P_k)}) = \max_{[Q]_{P_k} \in Q(P_k)} \mathrm{d}_{\mathcal{P}_g}(f(P_k), f([Q]_{P_k})).$$

We already observed that the distance function $\mathrm{d}_{\mathcal{P}_g}$ is invariant under $\mathrm{Hom}^+(\mathcal{S}_g)$, and we conclude that

$$b_P = \max_{[Q]_{P_k} \in Q(P_k)} \mathrm{d}_{\mathcal{P}_g}(P_k, [Q]_{P_k}) = b_{P_k} \leq \max_{1 \leq j \leq n} b_{P_j} = b,$$

which finishes the proof. □

The last ingredient tells us that any Weil-Petersson geodesic of length less than 1 can be covered by a finite chain of sub level sets. Most importantly, the length of this chain is independent of the geodesic.

Lemma 3.12. *Given $L > \mathrm{B}_g$, there exists an integer $J \geq 1$ with the following property: if $[X]$ and $[Y]$ are two points in $\mathrm{Teich}(\mathcal{S}_g)$ and $\{[X_t]\}_{t \in [0,1]}$ is a unit-speed WP-geodesic between $[X]$ and $[Y]$ of length one, then there exist pants decompositions $P_1, \dots, P_J$ so that $[X_t]$ lies in one of the sub level sets $V_L(P_j)$, $1 \leq j \leq J$, for any $t \in [0,1]$.*

Proof. Since the sub level sets form an open cover of $\mathrm{Teich}(\mathcal{S}_g)$ and since $\{[X_t]\}_{t\in[0,1]}$ is compact, we can pick finitely many pants decompositions $P_1,\ldots,P_m$ such that

$$\{[X_t]\}_{t\in[0,1]} \subset V(P_1)\cup\cdots\cup V(P_m).$$

We trivially have that $\{[X_t]\}_{t\in[0,1]}$ is also contained in $V_L(P_1)\cup\cdots\cup V_L(P_m)$, but, a priori, the index m depends on $[X]$ and $[Y]$. Given any pants decomposition P, consider the function

$$d_P\colon \partial V(P)\to\mathbb{R}_{\geq 0},\ [X]\mapsto \inf_{[Y]\in\partial V_L(P)} \mathrm{d}_{\mathrm{WP}}([X],[Y]).$$

We claim that there exists an $\epsilon>0$ depending only on g and L such that d_P takes no values in $(0,\epsilon)$. In other words, $\mathrm{distance}_{\mathrm{WP}}(V(P),\partial V_L(P))\geq\epsilon$. Suppose for now that the claim holds. Using geodesic convexity of the sub level sets, the claim implies that if $[X_t]\in V(P)$, then $[X_s]\in V_L(P)$ for any $s\in(t-\epsilon,t+\epsilon)$. Thus, if we set $J=\lceil 1/\epsilon\rceil$, then we can pick J pants decompositions $P_1,\ldots,P_J$ from the original $P_1,\ldots,P_m$ (possibly with repetition) such that

$$\{[X_t]\}_{t\in[0,1]} \subset V_L(P_1)\cup\cdots\cup V_L(P_J).$$

It remains to prove the claim. As in the previous proof, let $\mathrm{Tw}(P)$ denote the subgroup of $\mathrm{MCG}(\mathcal{S}_g)$ generated by Dehn twists along geodesics in P. We know that the Weil-Petersson metric extends to $\overline{\mathrm{Teich}(\mathcal{S}_g)}$ via its metric completion, so the map d_P also extends to the d_{WP}-metric completion $\overline{\partial V(P)}$ of $\partial V(P)$. We claim that the action of $\mathrm{Tw}(P)$ on $\overline{\mathrm{Teich}(\mathcal{S}_g)}$ is cocompact on $\overline{V_L(P)}$, meaning that there exists a compact set $K\subset\overline{V_L(P)}$ such that every element in $\overline{V_L(P)}/\mathrm{Tw}(P)$ has a lift in K (in particular, $\overline{V_L(P)}/\mathrm{Tw}(P)$ is compact). But this is quite obvious if we use P for Fenchel-Nielsen coordinates, since elements of $\mathrm{Tw}(P)$ leave the length coordinates, which are bounded by L, invariant and can be used to translate the twist coordinates into the compact set $[0,\pi]$. The metric completion of $\partial V_L(P)$ in extended Fenchel-Nielsen coordinates is the set with length coordinates in $[0,L]$ and at least one length coordinate exactly L. Hence, it can be written as follows:

$$\overline{\partial V_L(P)} = \left[\bigcap_{\gamma\in P}\left(L_{[X]}(\gamma)\Big|_{\overline{V_L(P)}}\right)^{-1}([0,L])\right]\bigcap\left[\bigcup_{\gamma\in P}\left(L_{[X]}(\gamma)\Big|_{\overline{V_L(P)}}\right)^{-1}(L)\right],$$

where $L_{[X]}(\gamma)$ is to be understood as the continuous map

$$\overline{\mathrm{Teich}(\mathcal{S}_g)}\to\mathbb{R}_{\geq 0},\ [X]\mapsto L_{[X]}(\gamma).$$

From this expression, it follows that $\overline{\partial V_L(P)}$ is relatively closed in $\overline{V_L(P)}$.[10] Therefore, $\overline{\partial V_L(P)}/\mathrm{Tw}(P)$ and $\overline{\partial V(P)}/\mathrm{Tw}(P)$ are disjoint compact subsets of $\overline{V_L(P)}/\mathrm{Tw}(P)$. Next, note that since the Weil-Petersson metric is invariant under the action of $\mathrm{MCG}(\mathcal{S}_g)$, we have

$$\begin{aligned}\inf_{\substack{[X]\in\partial V(P)\\ [Y]\in\partial V_L(P)}}\mathrm{d}_{\mathrm{WP}}([X],[Y]) &\geq \inf_{\substack{[X]\in\partial V(P)/\mathrm{Tw}(P)\\ [Y]\in\partial V_L(P)/\mathrm{Tw}(P)}}\mathrm{d}_{\mathrm{WP}}([X],[Y])\geq\\ \geq \inf_{\substack{[X]\in\overline{\partial V(P)}/\mathrm{Tw}(P)\\ [Y]\in\overline{\partial V_L(P)}/\mathrm{Tw}(P)}}\mathrm{d}_{\overline{\mathrm{WP}}}([X],[Y]) &= \min_{\substack{[X]\in\overline{\partial V(P)}/\mathrm{Tw}(P)\\ [Y]\in\overline{\partial V_L(P)}/\mathrm{Tw}(P)}}\mathrm{d}_{\overline{\mathrm{WP}}}([X],[Y])>0.\end{aligned}$$

In other words, the map

$$I\colon \mathcal{P}_gV\to\mathbb{R}_{\geq 0},\ P\mapsto \inf_{[X]\in\partial V(P)} d_P([X])$$

[10] This is not obvious since the over-line denotes metric completion, not necessarily closure.

is strictly positive. Note that it is also $\mathrm{MCG}(\mathcal{S}_g)$-invariant. Indeed, for $[f] \in \mathrm{MCG}(\mathcal{S}_g)$, we have the equalities of sets

$$V_L([f]\cdot P) = \left\{[X] \in \mathrm{Teich}(\mathcal{S}_g) \;\middle|\; \max_{\gamma\in[f]\cdot P} L_{[X]}(\gamma) < L\right\} = \\ = \left\{[X] \in \mathrm{Teich}(\mathcal{S}_g) \;\middle|\; \max_{\gamma\in P} L_{[f^{-1}]\cdot[X]}(\gamma) < L\right\} = [f](V_L(P))$$

and, hence,

$$I([f]\cdot P) = \inf_{\substack{[X]\in\partial V([f]\cdot P)\\ [Y]\in\partial V_L([f]\cdot P)}} \mathrm{d_{WP}}([X],[Y]) = \inf_{\substack{[X]\in[f]\cdot\partial V(P)\\ [Y]\in[f]\cdot\partial V_L(P)}} \mathrm{d_{WP}}([X],[Y]) = \\ = \inf_{\substack{[X]\in\partial V(P)\\ [Y]\in\partial V_L(P)}} \mathrm{d_{WP}}([f]\cdot[X],[f]\cdot[Y]) = \inf_{\substack{[X]\in\partial V(P)\\ [Y]\in\partial V_L(P)}} \mathrm{d_{WP}}([X],[Y]) = I(P).$$

Thus, I induces a map on $\mathcal{P}_g V/\mathrm{MCG}(\mathcal{S}_g)$, which is never zero. Since $\mathcal{P}_g V/\mathrm{MCG}(\mathcal{S}_g)$ is finite, we can pick ϵ to be the minimum value that the induced map achieves. This finishes the proof of the claim. □

We can now prove lemma 3.9, which also finishes the proof of theorem 3.7 and, hence, also concludes theorem 3.1.

Proof of lemma 3.9. Let P and P' be any two pants decompositions, and let $\{[X_t]\}_{t\in[0,T]}$ be a WP-geodesic between $\mathcal{Q}(P)$ and $\mathcal{Q}(P')$, $T = \mathrm{d_{WP}}(\mathcal{Q}(P),\mathcal{Q}(P'))$. For any pants decomposition Q, let $I(Q)$ denote the subset of $[0,1]$ with $[X_t] \in V_{2\mathrm{B}_g}(Q)$. By lemma 3.5, the set $I(Q)$ is always an interval. Apply the last lemma with $L = 2\mathrm{B}_g$ to get an integer $J \geq 1$ and pants decompositions $P_1,\dots,P_N$ such that $[X_t]$ lies in one of the sub level sets $V_L(P_j)$, $1 \leq j \leq N$, for any $t \in [0,1]$, and N is bounded by $J\cdot\lceil T\rceil$ (recall that one assumption of the lemma was that the geodesic has length one). Phrased in the new language, we have that $I(P_1)\cup\dots\cup I(P_N)$ covers the WP-geodesic, and, by relabeling, we may assume that the supremum of $I(P_j)$ is contained in $I(P_{j+1})$, for every $1 \leq j \leq N-1$. The inequality for N implies $N \leq J(T+1)$, i.e.

$$\frac{N}{J} - 1 \leq T = \mathrm{d_{WP}}(\mathcal{Q}(P),\mathcal{Q}(P')),$$

which is of the form we want if we can substitute $\mathrm{d}_{\mathcal{P}_g}(P,P')$ into the left hand side. The supremum property of the intervals $I(P_j)$ implies that each intersection $V_L(P_j)\cap V_L(P_{j+1})$, $1 \leq j \leq N-1$, is non-empty, which enables us to apply lemma 3.11. The latter yields a constant $b > 0$ with $\mathrm{d}_{\mathcal{P}_g}(P_j,P_{j+1}) < b$, for every $1 \leq j \leq N-1$. Moreover, the sub level sets $V_L(P)$ and $V_L(P_1)$ also intersect since $\mathcal{Q}(P)$ lies in both. The same holds for $V_L(P')$ and $V_L(P_N)$. By the triangle inequality, we get $\mathrm{d}_{\mathcal{P}_g}(P,P') \leq b(N+1)$. Combining all the inequalities yields

$$\mathrm{d_{WP}}(\mathcal{Q}(P),\mathcal{Q}(P')) \geq \frac{N}{J} - 1 \geq \frac{1}{J}\left(\frac{\mathrm{d}_{\mathcal{P}_g}(P,P')}{b} - 1\right) - 1 = \frac{\mathrm{d}_{\mathcal{P}_g}(P,P')}{Jb} - (\frac{1}{J}+1).$$

This finishes the proof because J and b depended only on $L = 2\mathrm{B}_g$, which depends only on g. □

A Preliminaries from Hyperbolic Geometry

We collect a few results from hyperbolic geometry needed for the paper. For each result, we give a reference where a proof can be found. We begin by recalling the famous Gauss-Bonnet formula. There are a lot of sources, where this result is proved. For example, one such source is [11, p. 164].

Theorem A.1 (Gauss-Bonnet). *Let X be a hyperbolic surface homeomorphic to $\mathcal{S}_g$, $g \geq 2$. Then* $\mathrm{Area}(X) = -2\pi\chi(X)$*, where $\chi(X) = 2 - 2g$ is the Euler characteristic of X.*

Next, we consider curves on a hyperbolic surface. The first theorem asserts the existence of a geodesic representative as well as its fundamental properties. This theorem is also a standard result, and one of many proofs can be found in [5, p. 23].

Theorem A.2. *Let c be an essential closed curve in $\mathcal{S}_{g,b}$. Then c is isotopic to a unique (not necessarily simple) geodesic γ, and γ is either contained in $\partial\mathcal{S}_{g,b}$ or $\gamma \cap \mathcal{S}_{g,b} = \emptyset$. Moreover, if c is simple, then so is γ. Lastly, if c is simple, then c and γ bound an embedded annulus (unless $c = \gamma$).*

We can use geodesics to embed disks and cylinders in a given hyperbolic surface. By the former, we refer to the notion of the injectivity radius. Let X be a hyperbolic surface and fix a point $p \in X$. The *injectivity radius $r_p(X)$ of X at p* is the supremum over all radii r for which the open ball of radius r around p is an isometrically embedded disk. The *injectivity radius $r_*(X)$ of X* is the infimum over all $r_p(X)$, $p \in X$. We have the following relation between injectivity radii and geodesics, see, for instance, [5, p. 96].

Proposition A.3. *Let X be a hyperbolic surface. Given $p \in X$, let γ_p be the shortest geodesic that runs through p. Then $r_p(X) = L_X(\gamma_p)/2$. Moreover, if γ is the shortest geodesic in X, then $r_*(X) = L_X(\gamma)/2$. In particular, $r_*(X) > 0$ as well as $r_p(X) > 0$, for all $p \in X$.*

Geodesics are related to embedded cylinders by the fundamental collar lemma. This first version for compact hyperbolic surfaces can be found in [5, p. 94].

Theorem A.4 (The Collar Lemma, compact case). *Fix some element $X \in \mathcal{M}(\mathcal{S}_g)$, $g \geq 2$, and let $\gamma_1, \ldots, \gamma_k$ be pairwise disjoint geodesics in X. Then we must have $k \leq 3g - 3$, and there exist geodesics $\gamma_{k+1}, \ldots, \gamma_{3g-3}$ such that $\gamma_1, \ldots, \gamma_{3g-3}$ form a pair of pants decomposition of X. The collars*

$$C(\gamma_j) = \left\{p \in X \;\middle|\; \mathrm{distance}(p, \gamma_j) \leq w(\gamma_j)\right\},$$

where $w(\gamma_j) = \mathrm{arcsinh}\left(\frac{1}{\sinh(L_X(\gamma_j)/2)}\right)$, are pairwise disjoint and each collar is isometric to the cylinder $[-w(\gamma_j), w(\gamma_j)] \times \mathcal{S}^1$ equipped with the Riemannian metric $ds^2 = d\rho^2 + L_X(\gamma_j)^2 \cosh^2(\rho) dt^2$.

The second version deals with non-compact surfaces of type $\mathcal{S}_{g,n}$ and can be found in [5, p. 112].

Theorem A.5 (The Collar Lemma, non-compact case). *Fix a hyperbolic surface X homeomorphic to $\mathcal{S}_{g,n}$, $2 - 2g - n < 0$, and let $\gamma_1, \ldots, \gamma_k$ be pairwise disjoint geodesics in X. Then we must have $k \leq 3g - 3 + n$, and there exist geodesics $\gamma_{k+1}, \ldots, \gamma_{3g-3+n}$ such that $\gamma_1, \ldots, \gamma_{3g-3+n}$ form a pair of pants decomposition of X. The collars*

$$C(\gamma_j) = \left\{p \in X \;\middle|\; \mathrm{distance}(p, \gamma_j) \leq w(\gamma_j)\right\},$$

where $w(\gamma_j) = \mathrm{arcsinh}\left(\frac{1}{\sinh(L_X(\gamma_j)/2)}\right)$, are pairwise disjoint and do not intersect the cusps of X.

Lastly, we can define the *curve complex* $\mathcal{C}(\mathcal{S})$ of a surface $\mathcal{S}$. We take the isotopy classes of geodesics as vertices and link two vertices by an edge if and only if their geometric intersection number is zero. A proof of connectivity of the curve complex is given in [6, p, 92].

Theorem A.6. *The curve complex $\mathcal{C}(\mathcal{S}_{g,n})$ of a surface of type $\mathcal{S}_{g,n}$ is connected, whenever $3g + n \geq 5$.*

Note that the hypothesis of this theorem are satisfied if and only if the genus of the surface is at least two or the surface is a torus with at least two punctures.

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