Currents in Geometry and Analysis

by
Florian Rossmannek

A Reading Course for the
Master of Science Mathematics

Supervised by Prof. Paul Biran
Co-supervised by Dr. Gleb Smirnov
Department of Mathematics, ETH Zürich
January 28, 2019

Abstract: After discussing basics of Sheaf theory and proving the de Rham theorem, we introduce currents and their associated cohomology, showing that it is isomorphic to the Dolbeault cohomology. We proceed to discuss exactness results related to the $\bar{\partial}$-operator and positive currents such as various Poincaré lemmata and the Poincaré Lelong equation, and we construct the Lelong number. In the last chapter, we prove the Levi Extension and the Proper Mapping Theorem.

Contents

1 Sheaf Theory **2**
1.1 Basic Concepts . . . 2
1.2 Cohomology of Sheaves . . . 6
1.3 The de Rham Theorem . . . 8

2 Currents **11**
2.1 Distributions and Currents . . . 11
2.1.1 Regularity of the Differential Operator . . . 15
2.1.2 Cohomology of Currents . . . 16
2.1.3 The Bochner-Martinelli Kernel . . . 17
2.2 Positive Currents . . . 19
2.2.1 Poincaré Lemmata . . . 19
2.2.2 The Poincaré Lelong equation . . . 22
2.2.3 The Lelong Number . . . 25
2.3 Application: Intersection of Analytic Subvarieties . . . 27

3 The Proper Mapping Theorem **28**
3.1 Divisors and Line Bundles . . . 28
3.2 The Levi Extension Theorem . . . 30
3.3 The Proof of the Proper Mapping Theorem . . . 33

Introduction

These notes were written during a reading course the author took at ETH Zürich in fall 2018. The aim was to introduce himself to currents and their applications in algebraic geometry, complex geometry, and complex analysis with primary use of Harris and Griffiths' book "Principles of Algebraic Geometry", [3]. The notes intend to rephrase some of the results and fill in some of the gaps left out in various proofs. The first chapter is concerned with Sheaf theory with the goal of proving the de Rham theorem as it is needed in the next chapter. The latter introduces currents, establishes their representation as differential forms with distributions as coefficients, and deduces the regularity of the $\bar{\partial}$-operator for currents. Next, their associated cohomology is constructed and proved to be isomorphic to the Dolbeault cohomology. Along the way, the Bochner-Martinelli kernel will be discussed. The second chapter continues with a survey of exactness results about positive currents. More precisely, we first prove then $\partial\bar{\partial}$-Poincaré lemma, then the Poincaré Lelong equation, and finally the Lelong number will be constructed. The chapter is finished with an application to the intersection number of analytic subvarieties. The last third of these notes aims at proving the Proper Mapping Theorem. To this end, a brief discussion of Divisors and Line Bundles is included, as well as the Levi Extension Theorem.

The first chapter is based on [3, p. 34ff] and [4, p. 60ff] with some additional inspiration from [6, p. 294ff]. The second one closely follows [3, p. 364ff]. An additional reference for some calculus on complex manifolds is [2, p. 101ff]. The section about the Bochner-Martinelli kernel is based on [5, p. 57ff]. The third chapter exclusively uses [3, p. 128ff,395ff].

1 Sheaf Theory

1.1 Basic Concepts

Let X be a topological space. We define a category $\mathcal{T}(X)$ as follows: the objects of $\mathcal{T}(X)$ are the open subsets of X; the morphism sets only contain the inclusion maps, i.e.

$$\mathrm{Hom}(U,V) = \begin{cases} \emptyset, & \text{if } U \not\subset V, \\ \{\iota_{U,V}\}, & \text{if } U \subset V. \end{cases}$$

A *presheaf* $\mathcal{F}$ *over* X is defined as a contravariant functor from $\mathcal{T}(X)$ to the category of abelian groups. Written out, this means that a preasheaf $\mathcal{F}$ over X is a collection of abelian groups $\mathcal{F}(U)$, one for each open subset $U \subset X$, together with a collection of group homomorphisms $r_{V,U}\colon \mathcal{F}(V) \to \mathcal{F}(U)$, one for each pair of inclusions $U \subset V$, that satisfy $r_{U,U} = \mathrm{id}_{\mathcal{F}(\mathrm{U})}$ and $r_{W,U} = r_{V,U} \circ r_{W,V}$, whenever $U \subset V \subset W$. We call elements in $\mathcal{F}(U)$ *sections over* U and the homomorphisms $r_{V,U}$ *restriction maps.* Given $\sigma \in \mathcal{F}(V)$, we will also use the notation $\sigma|_U$ to denote $r_{V,U}(\sigma)$.

A *sheaf over* X is defined to be a presheaf $\mathcal{F}$ over X with the following additional structure: if a section $\sigma \in \mathcal{F}(\bigcup_i U_i)$ satisfies $\sigma|_{U_i} = 0$ for all i, then we require $\sigma = 0$ in $\mathcal{F}(\bigcup_i U_i)$; moreover, given sections $\sigma_i \in \mathcal{F}(U_i)$ with $\sigma_i|_{U_i \cap U_j} = \sigma_j|_{U_i \cap U_j}$ for all i and j, there must exist a section $\rho \in \mathcal{F}(\bigcup_i U_i)$ with $\rho|_{U_i} = \sigma_i$ for all i. Note that ρ is unique by the first requirement.

A presheaf (sheaf) $\mathcal{F}$ over X is a *subpresheaf* (*subsheaf*) of another presheaf (sheaf) $\mathcal{G}$ over X if $\mathcal{F}(U)$ is a subgroup of $\mathcal{G}(U)$ for every open set $U \subset X$, and if the maps $r^{\mathcal{F}}_{V,U}$ are simply the maps $r^{\mathcal{G}}_{V,U}$ restricted to $\mathcal{F}(V)$.

A *morphism* (or *map*) bewteen two presheaves over the same space X simply is a natural transformation in the categorial sense. In other words, a morphism $\alpha\colon \mathcal{F} \to \mathcal{G}$ is a collection of group homomorphisms $\alpha_U\colon \mathcal{F}(U) \to \mathcal{G}(U)$ such that $r^{\mathcal{G}}_{V,U} \circ \alpha_V = \alpha_U \circ r^{\mathcal{F}}_{V,U}$, for all $U \subset V \subset X$. If $\mathcal{F}$ and $\mathcal{G}$ are two sheaves over X, then a *morphism of sheaves* is a morphism of the underlying presheaf structure. Naturally, *isomorphisms* between two presheaves (sheaves) over X are natural isomorphisms in the categorial sense.

Note that the commutativty condition in the definition of a morphism $\alpha\colon \mathcal{F} \to \mathcal{G}$ immediately implies that the maps $r^{\mathcal{F}}_{V,U}$ restrict to well-defined maps $\ker(\alpha_V) \to \ker(\alpha_U)$. Thus, there is a well-defined presheaf $\ker(\alpha)$. The same argument gives us presheaves $\mathrm{coker}(\alpha)$ and $\mathrm{im}(\alpha)$. If α is a morphism of sheaves, then $\ker(\alpha)$ actually is a sheaf itself.

Lemma 1.1. *Given a morphism $\alpha\colon \mathcal{F} \to \mathcal{G}$ of sheaves over a topological space X, $\ker(\alpha)$ is a sheaf.*

Proof. We only need to check that $\ker(\alpha)$ satisfies the additional structure of a sheaf over a presheaf. Suppose $V = \bigcup_i U_i$. The first property is clear, because $\mathcal{F}$ is a sheaf, and if σ is zero in $\ker(\alpha)(V)$, then it is also zero in $\mathcal{F}(V)$ and, hence, each $\sigma|_{U_i}$ is zero. Now suppose we are given sections $\sigma_i \in \ker(\alpha)(U_i)$ with $\sigma_i|_{U_i \cap U_j} = \sigma_j|_{U_i \cap U_j}$. Then there is a section $\rho \in \mathcal{F}(V)$ with $\rho|_{U_i} = \sigma_i$. Further, since

$$r^{\mathcal{G}}_{V,U_i} \circ \alpha_V(\rho) = \alpha_{U_i} \circ r^{\mathcal{F}}_{V,U_i}(\rho) = \alpha_{U_i}(\sigma_i) = 0,$$

the sheaf property of $\mathcal{G}$ implies $\alpha_V(\rho) = 0$, proving that ρ is not just an element of $\mathcal{F}(V)$, but of $\ker(\alpha)(V)$. □

However, $\mathrm{coker}(\alpha)$ and $\mathrm{im}(\alpha)$ are *not* necessarily sheaves. We refer to the next section for an example of this. We would like to circumvent this inconveniance and somehow turn $\mathrm{coker}(\alpha)$ and $\mathrm{im}(\alpha)$ into sheaves. To this end, we need to introduce a new object. Fix a point $p \in X$ and denote by I_p the set of all open sets containing p. We can turn I_p into a directed set by specifying $U \geq V$ if $U \subset V$. Now let $\mathcal{F}$ be a presheaf over X. By the properties of the restriction maps, $(\mathcal{F}(V), r_{V,U})$, $U, V \in I_p$, is a directed system over I_p. As such, we can take the direct limit to obtain the so-called *stalk of $\mathcal{F}$ at p*,

$$\mathcal{F}_p = \varinjlim \mathcal{F}(U) = \bigsqcup_{U \in I_p} \mathcal{F}(U)/\sim .$$

Thus, the stalk contains equivlance classes $[\sigma]$, $\sigma \in \mathcal{F}(U)$, where $\sigma \in \mathcal{F}(U)$ and $\tau \in \mathcal{F}(V)$ lie in the same equivalence class if there is an open set $p \in W \subset U \cap V$ such that $\sigma|_W = \tau|_W$. The stalk can be of interest in its own right, but for now we merely use it for further constructions. Given an open subset $U \subset X$, set

$$\mathcal{F}^*(U) = \left\{ s\colon U \to \bigsqcup_{p \in U} \mathcal{F}_p \;\middle|\; \begin{array}{l} (1)\ \forall p \in U\colon s(p) \in \mathcal{F}_p, \\ (2)\ \forall p \in U\ \exists p \in V \subset U\ \exists \tau \in \mathcal{F}(V)\ \forall q \in V\colon s(q) = [\tau] \end{array} \right\}.$$

Lemma 1.2. *$\mathcal{F}^*$ together with the group operation $s \circ t$ defined pointwise by $s \circ t(p) = [s(p) \circ_{\mathcal{F}(U)} t(p)]$ and the obvious restriction maps form a sheaf over X. Moreover, there exists a morphism $\theta\colon \mathcal{F} \to \mathcal{F}^*$ such that for all sheaves $\mathcal{G}$ over X and for all morphisms $\alpha\colon \mathcal{F} \to \mathcal{G}$ there exists a unique morphism $\alpha^*\colon \mathcal{F}^* \to \mathcal{G}$ with $\alpha = \alpha^* \circ \theta$.*

Proof. First note that the obvious restriction maps are well-defined exactly by property (1). Hence, $\mathcal{F}^*$ certainly is a presheaf. The sheaf conditions follow just as quickly: If the restriction of $s \in \mathcal{F}^*(\bigcup_i U_i)$ to U_i is zero, then, $s(p) = s|_{U_i}(p) = 0$ for any $p \in U_i$. This holds for all i so that $s = 0$ in $\mathcal{F}^*(\bigcup_i U_i)$. Lastly, if we are given $s_i \in \mathcal{F}^*(U_i)$ that agree on each intersection, then we construct the needed ρ in the obvious way. Next, define the morphism θ to be "the constant map",

$$\theta_U\colon \mathcal{F}(U) \to \mathcal{F}^*(U),\ \theta_U(\sigma) = (p \mapsto [\sigma]).$$

It remains to prove the existence of the morphisms α^*. Let $\alpha\colon \mathcal{F} \to \mathcal{G}$, $U \subset X$, and $s \in \mathcal{F}^*(U)$ be given. By the second property in $\mathcal{F}^*$, for every point $p \in U$ there is a neighbourhood $V_p \subset U$ and an element $\tau_p \in \mathcal{F}(V_p)$ with $s|_{V_p} = \theta_{V_p}(\tau_p)$. We know that

$$\theta_{V_p \cap V_q}\left(\tau_p\big|_{V_p \cap V_q} - \tau_q\big|_{V_p \cap V_q}\right) = \left(s|_{V_p}\right)\big|_{V_p \cap V_q} - \left(s|_{V_q}\right)\big|_{V_p \cap V_q} = 0.$$

This means that $\tau_p - \tau_q$ is zero in a small neighbourhood W_r of any point r in $V_p \cap V_q$. In particular, for any point r in $V_p \cap V_q$ we get

$$\alpha_{V_p}(\tau_p)\big|_{W_r} - \alpha_{V_q}(\tau_q)\big|_{W_r} = 0.$$

By the sheaf property of $\mathcal{G}$, $\alpha_{V_p}(\tau_p)$ and $\alpha_{V_q}(\tau_q)$ agree on $V_p \cap V_q$. Using the sheaf property of $\mathcal{G}$ again, we conclude that there is a section $\rho \in \mathcal{G}(U)$ that agrees with $\alpha_{V_p}(\tau_p)$ on V_p. Now we set $\alpha^*_U(s) = \rho$. This is well-defined because for a different choice of neighborhoods $\tilde{V}_p$ and sections $\tilde{\tau}_p$, we still have $\tau_p = \tilde{\tau}_p$ on a small neighborhood of p and, hence, $\rho = \tilde{\rho}$ on this neighborhood. By the sheaf property of $\mathcal{G}$, we get $\rho = \tilde{\rho}$. This α^* is a morphism of sheaves since

$$\alpha^*_V(s)\big|_U = \rho\big|_U = \alpha^*_U(s|_U)$$

for any pair $U \subset V$. Clearly, we have $\alpha^*_U \circ \theta_U(\sigma) = \alpha_U(\sigma)$ for any section $\sigma \in \mathcal{F}(U)$, since in this case we simply take $\tau_p = \sigma|_{V_p}$. To prove uniqueness, suppose there was a another morphism $\beta\colon \mathcal{F}^* \to \mathcal{G}$ with $\alpha = \beta \circ \theta$. Fix any open set $U \subset X$ and a section $s \in \mathcal{F}^*(U)$. Any point in U has a small neighborhood V in which $s|_V \equiv [\tau]$ for some $\tau \in \mathcal{F}(V)$. Then

$$\alpha^*_U(s)\big|_V = \alpha^*_V \circ \theta_V(\tau) = \alpha_V(\tau) = \beta_V \circ \theta_V(\tau) = \beta_U(s)\big|_V$$

and, hence, $\alpha^*_U(s) = \beta_U(s)$ by the sheaf property of $\mathcal{G}$. $\square$

Note that the lemma describes a universal property for $(\mathcal{F}^*, \theta)$. Thus, as a formal consequence, $\mathcal{F}^*$ is unique up to unique isomorphisms. We call $\mathcal{F}^*$ the *sheaf associated to the presheaf* $\mathcal{F}$. Property (2) in the definition of $\mathcal{F}^*(U)$ immediately implies that the stalk of $\mathcal{F}^*$ at any point $p \in X$ is exactly the stalk of $\mathcal{F}$ at p. This observation together with the next proposition below prove that if $\mathcal{F}$ was a sheaf to begin with, then $\mathcal{F}^*$ is isomorphic to $\mathcal{F}$ via θ. To make sense of the statement of the next proposition, observe the following: Let $\alpha\colon \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves over X. The fact that the morphism commutes with the restriction maps, by definition, implies that the morphism induces well-defined group homomorphisms $\alpha_p\colon \mathcal{F}_p \to \mathcal{G}_p$ on the stalks by $[\sigma] \mapsto [\alpha|_U(\sigma)]$.

Proposition 1.3. *$\alpha\colon \mathcal{F} \to \mathcal{G}$ is an isomorphism of sheaves over X if and only if for every $p \in X$ the induced map $\alpha_p\colon \mathcal{F}_p \to \mathcal{G}_p$ is a group isomorphism.*

Proof. If α is an isomorphism, then the induced map on the stalks $(\alpha^{-1})_p$ clearly is an inverse map for α_p. Now assume that each map α_p is an isomorphism. For each $U \subset X$, we will show that $\alpha_U : \mathcal{F}(U) \to \mathcal{G}(U)$ is a group isomorphism. Then the morphism $\beta\colon \mathcal{G} \to \mathcal{F}$ defined by $\beta_U = (\alpha_U)^{-1}$ is an inverse of α as morphisms of sheaves.

For injectivity, suppose $\alpha_U(\sigma) = 0$, where $\sigma \in \mathcal{F}(U)$. Then we have $\alpha_p([\sigma]_p) = 0$ in $\mathcal{G}_p$ for every $p \in U$. By hypothesis, each α_p is injective so that $[\sigma]_p$ is zero in $\mathcal{F}_p$ for all points $p \in U$. By definition, this means that σ restricted to a small neighborhood of $p \in U$ is identically zero. Since this holds for all p, the glueing property of the sheaf $\mathcal{F}$ asserts that $\sigma = 0$ in $\mathcal{F}(U)$.

For surjectivity, let $\tau \in \mathcal{G}(U)$ be given. By assumption, $[\tau]_p \in \mathcal{G}_p$ has an inverse image $S_p \in \mathcal{F}_p$ under α_p, for every $p \in U$. Write each S_p as the equivalence class of some section $\sigma_p \in \mathcal{F}(V_p)$, where V_p is some small neigbourhood of p. Then, by construction, $\alpha_{V_p}(\sigma_p)$ and τ agree on a (possibly shrunken) neighborhood V_p of p. For two different points p and q in U, we have

$$\alpha_{V_p}(\sigma_p)\Big|_{V_p \cap V_q} = \tau\Big|_{V_p \cap V_q} = \alpha_{V_q}(\sigma_q)\Big|_{V_p \cap V_q}.$$

By injectivity of $\alpha_{V_p \cap V_q}$ proved in the first half, σ_p and σ_q agree on $V_p \cap V_q$. By the sheaf property of $\mathcal{F}$, we can glue all the sections σ_p to some $\sigma \in \mathcal{F}(U)$. This section satisfies $\alpha_U(\sigma)|_{V_p} = \tau|_{V_p}$ so that, by the sheaf property of $\mathcal{G}$, the sections $\alpha_U(\sigma)$ and τ agree on all of U. $\square$

Having defined the sheaf associated to a presheaf, we can now speak of the cokernel and the image of a morphism $\alpha\colon \mathcal{F} \to \mathcal{G}$. To avoid confusion, from now on, we will denote the actual presheaf cokernel and image by p-coker(α) and p-im(α), respectively, and write coker(α) and im(α) for the sheaves associated to these presheaves. Furthermore, we can also define the *quotient sheaf* $\mathcal{F}/\mathcal{F}'$ of $\mathcal{F}$ by a subsheaf $\mathcal{F}'$ as the sheaf associated to the obvious presheaf $\mathcal{F}(U)/\mathcal{F}'(U)$, $U \subset X$.

We say that a morphism $\alpha\colon \mathcal{F} \to \mathcal{G}$ of sheaves over X is *injective* if $\ker(\alpha) = 0$, i.e. if, for every open set $U \subset X$, $\alpha_U\colon \mathcal{F}(U) \to \mathcal{G}(U)$ is injective. Observe that the morphism $\alpha^*\colon \mathcal{F}^* \to \mathcal{G}$ from lemma 1.2 is injective whenever α is. In particular, if we take our morphism to be the inclusion of p-im(α) into $\mathcal{G}$, then we get a unique injective map from im(α) into $\mathcal{G}$. Hence, we can consider im(α) as a subsheaf of $\mathcal{G}$ and it makes sense to call α *surjective* if im$(\alpha) = \mathcal{G}$. Likewise, we can view coker(α) as a subsheaf of $\mathcal{F}$.

Let us point out an important remark. Strictly speaking, we defined the notions of injectivity and surjectivity only for morphisms between sheaves, not for presheaves. While this would be no problem for injectivity, there is an ambiguity for the definition of surjectivtiy for presheaves, because requiring im$(\alpha) = \mathcal{G}$ is something different than requiring p-im$(\alpha) = \mathcal{G}$. For the sake of simplicity, we mean sheaves whenever we talk about surjective morphisms. This argument also shows that surjectivity of a morphism α between sheaves is not equivalent to surjectivity of each map α_U.

Next, we would like to show that the usual equivalence between being bijective and being an isomorphism holds. The way we set up the definitions, this is not a trivial statement. Begin by observing that the stalk of $\ker(\alpha)$ at a point $p \in X$ is exactly the kernel of the group homomorphism α_p induced on the stalks. The same goes for the image as

$$(\mathrm{im}(\alpha))_p = (\text{p-im}(\alpha))_p^* = (\text{p-im}(\alpha))_p = \mathrm{im}(\alpha_p).$$

The following lemma is an application of proposition 1.3.

Lemma 1.4. *A morphism $\alpha\colon \mathcal{F} \to \mathcal{G}$ of sheaves over X is injective (surjective) if and only if for every $p \in X$ the induced map $\alpha_p\colon \mathcal{F}_p \to \mathcal{G}_p$ on the stalk is injective (surjective).*

Proof. Suppose α is injective and $\alpha_p([\sigma]) = \alpha_p([\tau])$, $\sigma \in \mathcal{F}(U)$ and $\tau \in \mathcal{F}(V)$. The latter is equivalent to the existence of a smaller set $W \subset U \cap V$ with $\alpha_U(\sigma)|_W = \alpha_V(\tau)|_W$. By injectivitiy of α, we get $\sigma|_W = \tau|_W$ and, hence, $[\sigma] = [\tau]$ in $\mathcal{F}_p$. Conversely, if for every p the map $\alpha_p\colon \mathcal{F}_p \to \mathcal{G}_p$ is injective, then the maps $\alpha_p\colon \mathcal{F}_p \to \mathrm{im}(\alpha_p) = (\mathrm{im}(\alpha))_p$ are isomorphisms. By proposition 1.3, α viewed as a morphism $\mathcal{F} \to \mathrm{im}(\alpha)$ is an isomorphism and, in particular, each α_U is injective. For surjectivity, let i^* denote the injective morphism $\mathrm{im}(\alpha) \to \mathcal{G}$, which we used to identify $\mathrm{im}(\alpha)$ as a subsheaf of $\mathcal{G}$. α is surjective if and only if every $i_U^*\colon \mathrm{im}(\alpha)(U) \to \mathcal{G}(U)$ is surjective. As we know that i^* is injective, this is the case if and only if i^* is an isomorphism. By proposition 1.3, i^* is an isomorphism if and only if for every $p \in X$ the map $i_p^*\colon \mathrm{im}(\alpha_p) \to \mathcal{G}_p$ is one. As we know from the first half of the proof that the maps i_p^* are injective, they are isomorphisms if and only if they are surjective. This, in turn, is the case if and only if α_p is surjective, which finishes the proof. □

Using proposition 1.3 once more, this lemma shows that a morphism of sheaves is an isomorphism if and only if it is both injective and surjective. At first glance, this may come as a slight surprise: α is an isomorphism if and only if every α_U is bijective. However, we remarked earlier that surjectivity of α does not require all the maps α_U to be surjective. This is explained by the injectivity. The proof of the previous lemma reveals that if α is injective, then α as a morphism $\mathcal{F} \to \mathrm{im}(\alpha)$ is an isomorphism. In particular, we get $\mathrm{im}(\alpha)(U) = \mathrm{im}(\alpha_U) = \text{p-im}(\alpha)(U)$. In other words, when we have injectivity, then we get for free that the presheaf image already is a sheaf. The next lemma shows that the notions of kernel, cokernel, image, and quotient, for sheaves behave just as the same notions for other algebraic objects, say groups.

Lemma 1.5. *Given a morphism $\alpha\colon \mathcal{F} \to \mathcal{G}$ of sheaves over X,* $\mathrm{im}(\alpha)$ *is isomorphic to the quotient* $\mathcal{F}/\ker(\alpha)$ *and* $\mathrm{coker}(\alpha)$ *is isomorphic to* $\mathcal{G}/\mathrm{im}(\alpha)$.

Proof. As the stalk of a sheaf associated to a presheaf is exactly the stalk of the presheaf, the stalk of a quotient is just the quotient of the stalks. By lemma 1.4, the induced map $\alpha\colon \mathcal{F}/\ker(\alpha) \to \mathrm{im}(\alpha)$ is an isomorphism if and only if for every $p \in X$, the map

$$\alpha_p\colon \mathcal{F}_p/\ker(\alpha_p) = (\mathcal{F}/\ker(\alpha))_p \to \mathrm{im}(\alpha)_p = \mathrm{im}(\alpha_p)$$

is an isomorphism, which is obviously true. The same goes for the cokernel. □

Keeping both this lemma as well as lemma 1.4 in mind, we can now turn to exact sequences of sheaves and treat them similarly as exact sequences of groups. An *exact sequence of sheaves* is a chain of maps

$$\dots \to \mathcal{F}^{n-1} \xrightarrow{\alpha^{n-1}} \mathcal{F}^n \xrightarrow{\alpha^n} \mathcal{F}^{n+1} \to \dots$$

satisfying $\ker(\alpha^n) = \mathrm{im}(\alpha^{n-1})$ for every $n \in \mathbb{Z}$. Caution is appropiate when thinking of the local version of a short exaxct sequence,

$$0 \to \mathcal{E}(U) \xrightarrow{\alpha_U} \mathcal{F}(U) \xrightarrow{\beta_U} \mathcal{G}(U) \to 0,$$

because, as noted earlier, surjectivtiy of β as a morphism between sheaves does not necessarily imply surjectivity of β_U.

Examples

Some examples to have in mind are sheaves on manifolds given by smooth functions with addition, differential forms with addition, or closed differential forms. If the manifold is complex, then we can also consider holomorphic functions, differential forms of split type (p,q), or $\bar{\partial}$-closed differential forms. Let us review one example in more detail. This will be a counterexample to the fact that the presheaf cokernel and presheaf image are not automatically sheafs. Let $\mathcal{O}$ be the sheaf of holomorphic funtions with addition and $\mathcal{O}^*$ the sheaf of non-vanishing holomorphic funtions with multiplication on $\mathbb{C}^*$. Consider the morphism $\exp\colon \mathcal{O} \to \mathcal{O}^*$ that sends a holomorphic function $f \in \mathcal{O}(U)$, $U \subset \mathbb{C}^*$, to $e^{2\pi i f} \in \mathcal{O}^*(U)$. Due to the nature of the complex logarithm, the holomorphic funtion $g(z) = z$ in $\mathcal{O}^*$ is not the image of any $f \in \mathcal{O}$, but given any contractible set $U \subset \mathbb{C}^*$, g is in the image of $\mathcal{O}(U)$ under exp.

1.2 Cohomology of Sheaves

Fix a manifold M, a locally finite open cover $\underline{U} = \{U_i\}_{i\in I}$ of M, and a sheaf $\mathcal{F}$ over M. Let I_n denote the set of $(n+1)$-tuples of indices in I. We define a cochain complex by

$$\tilde{\mathrm{C}}^n(\underline{U},\mathcal{F}) = \prod_{(i_0,\dots,i_n)\in I_n} \mathcal{F}(U_{i_0}\cap\cdots\cap U_{i_n})$$

with coboundary operator $\delta\colon \tilde{\mathrm{C}}^n(\underline{U},\mathcal{F}) \to \tilde{\mathrm{C}}^{n+1}(\underline{U},\mathcal{F})$ given by

$$(\delta\sigma)_{(i_0,\dots,i_{n+1})} = \sum_{j=0}^{n+1} (-1)^j \sigma_{(i_0,\dots,\hat{i}_j,\dots,i_{n+1})}\Big|_{U_{i_0}\cap\cdots\cap U_{i_{n+1}}}.$$

That δ^2 is really 0 is a straight forward calculation,

$$\begin{aligned}
(\delta^2\sigma)_{(i_0,\dots,i_{n+2})} &= \sum_{k=0}^{n+2} (-1)^k (\delta\sigma)_{(i_0,\dots,\hat{i}_k,\dots,i_{n+2})}\Big|_{U_{i_0}\cap\cdots\cap U_{i_{n+2}}} \\
&= \sum_{k=0}^{n+2} (-1)^k \sum_{j<k} (-1)^j \sigma_{(i_0,\dots,\hat{i}_j,\dots,\hat{i}_k,\dots,i_{n+2})}\Big|_{U_{i_0}\cap\cdots\cap U_{i_{n+2}}} \\
&+ \sum_{k=0}^{n+2} (-1)^k \sum_{j>k} (-1)^{j-1} \sigma_{(i_0,\dots,\hat{i}_k,\dots,\hat{i}_j,\dots,i_{n+2})}\Big|_{U_{i_0}\cap\cdots\cap U_{i_{n+2}}} \\
&= 0.
\end{aligned}$$

Then, we let $\mathrm{C}^*(\underline{U},\mathcal{F})$ denote the subcomplex of alternating cochains, i.e. cochains σ with $\sigma_{a(i_0,\dots,i_n)} = \mathrm{sign}(a)\sigma_{(i_0,\dots,i_n)}$ for any permutation a. In particular, a cochain is zero whenever $i_j = i_k$ for some $j \neq k$. As usual, we consider the set of cocycles $\mathrm{Z}^n(\underline{U},\mathcal{F})$ and the set of coboundaries $\delta(\mathrm{Z}^{n-1}(\underline{U},\mathcal{F}))$ and define the cohomology groups $\mathrm{H}^n(\underline{U},\mathcal{F})$ as the quotient of these two.[1] We would like to contruct a homology independent of $\underline{U}$. This will be done by taking a direct limit, where the directed system is given as follows. Another locally finite open cover $\underline{U}' = \{U'_j\}_{j\in J}$ is said to be a *refinement* of $\underline{U}$ if there is a map $\phi\colon J \to I$ with $U'_j \subset U_{\phi(j)}$ for all $j \in J$. Such ϕ induces a map $\rho_\phi\colon \mathrm{C}^*(\underline{U},\mathcal{F}) \to \mathrm{C}^*(\underline{U}',\mathcal{F})$ via

$$(\rho_\phi\sigma)_{(j_0,\dots,j_n)} = \sigma_{(\phi(j_0),\dots,\phi(j_n))}\Big|_{U'_{j_0}\cap\cdots\cap U'_{j_n}},$$

[1] It is known that the resulting homology is isomorphic to the homology of $\tilde{\mathrm{C}}^*(\underline{U},\mathcal{F})$.

which clearly satisfies $\delta' \circ \rho_\phi = \rho_\phi \circ \delta$ and, hence, descends to a map on cohomology. We claim that for a different choice $\psi\colon J \to I$, for which $\underline{U}'$ also is a refinement of $\underline{U}$, the induced maps ρ_ϕ and ρ_ψ are chain homotopic. Indeed, a chain homotopy $\delta' \circ \chi + \chi \circ \delta = \rho_\phi - \rho_\psi$ is given by

$$\chi\colon \mathrm{C}^{n+1}(\underline{U}, \mathcal{F}) \to \mathrm{C}^n(\underline{U}', \mathcal{F}),\ (\chi\sigma)_{(j_0,\dots,j_n)} = \sum_{k=0}^{n} (-1)^k \sigma_{(\phi(j_0),\dots,\phi(j_k),\psi(j_k),\dots,\psi(j_n))}.$$

Thus, we can write $\rho\colon \mathrm{H}^*(\underline{U}, \mathcal{F}) \to \mathrm{H}^*(\underline{U}', \mathcal{F})$ independently of the choice of function $J \to I$. With this setup, we got a directed set with respect to refinement of covers and can define the *Čech cohomology of the sheaf* $\mathcal{F}$ as the direct limit

$$\check{\mathrm{H}}^*(M, \mathcal{F}) = \varinjlim \mathrm{H}^*(\underline{U}, \mathcal{F}).$$

We can easily calculate the zero-th Čech cohomology group.

Proposition 1.6. $\check{\mathrm{H}}^0(M, \mathcal{F}) = \mathcal{F}(M)$.

Proof. Given $\sigma = \{\sigma_i\}_{i \in I} \in \mathrm{C}^0(\underline{U}, \mathcal{F})$, its coboundary is

$$\delta\sigma = \left\{ \sigma_j\Big|_{U_i \cap U_j} - \sigma_i\Big|_{U_i \cap U_j} \right\}_{i \neq j}.$$

Thus, if σ is a cocycle, then σ_i and σ_j agree on $U_i \cap U_j$. By the property of a sheaf, we have $\sigma_i = \rho|_{U_i}$ for some $\rho \in \mathcal{F}(M)$. This proves that $\mathrm{H}^0(\underline{U}, \mathcal{F}) = \mathrm{Z}^0(\underline{U}, \mathcal{F}) = \mathcal{F}(M)$ for any locally finite cover $\underline{U}$. In particular,

$$\check{\mathrm{H}}^0(M, \mathcal{F}) = \varinjlim \mathrm{H}^0(\underline{U}, \mathcal{F}) = \mathcal{F}(M).$$

□

For a particular type of sheaf, we can also calculate the other cohomology groups in degree greater than zero. We say that a sheaf $\mathcal{F}$ is *fine* if it admits a partition of unity for any locally finite open cover $\underline{U}$ of M in the following sense: for every $j \in I$ and every open set U in the cover, there is a group homomorphism $\eta_{j,U}\colon \mathcal{F}(U \cap U_j) \to \mathcal{F}(U)$ such that, firstly, for any inclusion $U \subset V$, we have

$$r_{V,U} \circ \eta_{j,V} = \eta_{j,U} \circ r_{V \cap U_j, U \cap U_j}$$

and, secondly, for any section $\sigma \in \mathcal{F}(U)$,

$$\sum_{j \in I} \eta_{j,U} \left(\sigma\Big|_{U \cap U_j} \right) = \sigma.$$

Proposition 1.7. *For a fine sheaf $\mathcal{F}$ over M, $\check{\mathrm{H}}^n(M, \mathcal{F}) = 0$ for all $n \geq 1$.*

Proof. Let a cocycle $\sigma \in \mathrm{Z}^n(\underline{U}, \mathcal{F})$ be given. To shorten notation, abbreviate $U_n = U_{i_0} \cap \dots \cap U_{i_n}$ and write U_n^k for $U_{i_0} \cap \dots \cap \hat{U}_{i_k} \cap \dots \cap U_{i_n}$. Define an element $\tau \in \mathrm{C}^{n-1}(\underline{U}, \mathcal{F})$ by

$$\tau_{(i_0,\dots,i_{n-1})} = \sum_{j \in I} \eta_{j,U_{n-1}} \left(\sigma_{(j,i_0,\dots,i_{n-1})} \right).$$

We can compute

$$\sum_{k=0}^{n} (-1)^k \eta_{j,U_n^k} \left(\sigma_{(j,i_0,\dots,\hat{i}_k,\dots,i_n)} \right)\Big|_{U_n} = \eta_{j,U_n} \underbrace{\left(\sum_{k=0}^{n} (-1)^k \sigma_{(j,i_0,\dots,\hat{i}_k,\dots,i_n)}\Big|_{U_n \cap U_j} \right)}_{= \sigma_{(i_0,\dots,i_n)}|_{U_n \cap U_j} \text{ because } \delta\sigma = 0}$$

and, hence,

$$(\delta\tau)_{(i_0,\dots,i_n)} = \sum_{k=0}^{n}(-1)^k \sum_{j\in I} \eta_{j,U_n^k}\left(\sigma_{(j,i_0,\dots,\hat{i}_k,\dots,i_n)}\right)\Big|_{U_n}$$
$$= \sum_{j\in I} \eta_{j,U_n}\left(\sigma_{(i_0,\dots,i_n)}\Big|_{U_n\cap U_j}\right) = \sigma_{(i_0,\dots,i_n)}.$$

This proves that every cocycle is exact in $\mathrm{H}^n(\underline{U},\mathcal{F})$. Of course, the direct limit is then zero, as well. □

An example of a fine sheaf is the sheaf of smooth functions or the sheaf of differential forms, where a partition of unity is exactly that in the usual sense.

Our next goal is to prove the de Rham theorem. To this end, we first need to find a long exact sequence in Čech cohomology.

Proposition 1.8. *Suppose we have a short exact sequence of sheaves*

$$0 \to \mathcal{E} \xrightarrow{\alpha} \mathcal{F} \xrightarrow{\beta} \mathcal{G} \to 0.$$

Then there is a long exact sequence

$$\cdots \to \check{\mathrm{H}}^{n-1}(M,\mathcal{G}) \to \check{\mathrm{H}}^{n}(M,\mathcal{E}) \to \check{\mathrm{H}}^{n}(M,\mathcal{F}) \to \check{\mathrm{H}}^{n}(M,\mathcal{G}) \to \check{\mathrm{H}}^{n+1}(M,\mathcal{E}) \to \dots$$

Outline of Proof. Due to the surjectivity issue of $\beta_U : \mathcal{F}(U) \to \mathcal{G}(U)$, the sequence

$$0 \to \mathcal{E}(U) \xrightarrow{\alpha_U} \mathcal{F}(U) \xrightarrow{\beta_U} \mathcal{G}(U) \to 0$$

is not neccessarily exact. This is resolved by working with presheaves. Let $\mathcal{D}$ be the quotient *pre*sheaf $\mathcal{F}/\mathcal{E}$, where we consider $\mathcal{E}$ as a subsheaf of $\mathcal{F}$ via the injective morphism α. We did not define several notions for presheaves to avoid confusion, but one can verify that we can build an analogue cochain complex for a presheaf and that we get an exact sequence

$$0 \to \mathrm{C}^n(\underline{U},\mathcal{E}) \xrightarrow{\alpha} \mathrm{C}^n(\underline{U},\mathcal{F}) \xrightarrow{\beta} \mathrm{C}^n(\underline{U},\mathcal{D}) \to 0$$

for any open cover $\underline{U}$. Since this is just a cochain complex, we know that there exists a corresponding long exact sequence in cohomology. Thus, passing to the direct limit gives us a long exact sequence

$$\cdots \to \check{\mathrm{H}}^{n-1}(M,\mathcal{D}) \to \check{\mathrm{H}}^{n}(M,\mathcal{E}) \to \check{\mathrm{H}}^{n}(M,\mathcal{F}) \to \check{\mathrm{H}}^{n}(M,\mathcal{D}) \to \check{\mathrm{H}}^{n+1}(M,\mathcal{E}) \to \dots$$

Lastly, it is a technical result that $\check{\mathrm{H}}^*(M,\mathcal{D})$ is isomorphic to the cohomology of the sheaf $\mathcal{D}^*$ associated to the presheaf $\mathcal{D}$. The latter is exactly $\mathcal{G}$ by lemma 1.5 and exactness of the sequence of sheaves,

$$\mathcal{G} = \mathrm{im}(\beta) \cong \mathcal{F}/\ker(\beta) = \mathcal{F}/\mathrm{im}(\alpha) \cong \mathcal{F}/\mathcal{E} = \mathcal{D}^*.$$

□

1.3 The de Rham Theorem

Theorem 1.9 (de Rham)**.** *The singular cohomology* $\mathrm{H}^*_{\mathrm{sing}}(M,\mathbb{R})$ *with coefficients in* $\mathbb{R}$ *is isomorphic to the de Rham cohomology* $\mathrm{H}^*_{\mathrm{dR}}(M)$.

We will prove this theorem in two steps by showing that each of these cohomology theories is isomorphic to the Čech cohomology of a sheaf over M. Let $\mathbb{R}_M$ denote the sheaf of constant $\mathbb{R}$-valued functions.

Lemma 1.10. $\mathrm{H}^*_{\mathrm{sing}}(M,\mathbb{R})$ *is isomorphic to* $\check{\mathrm{H}}^*(M,\mathbb{R}_M)$.

Proof. By the universal coefficients theorem[2], it suffices to prove that $\mathrm{H}^*_{\mathrm{sing}}(M,\mathbb{Z})$ is isomorphic to $\check{\mathrm{H}}^*(M,\mathbb{Z}_M)$, where $\mathbb{Z}_M$ denotes the sheaf of constant $\mathbb{Z}$-valued functions. Let K denote the simplicial complex realizing the underlying topological space of M. It is a standard result that the simplicial cohomology of K is isomorphic to the singular cohomology of M. Therefore, it suffices to find an isomorphism from $\check{\mathrm{H}}^*(M,\mathbb{Z}_M)$ to $\mathrm{H}^*_{\mathrm{CW}}(K,\mathbb{Z})$. Given a vertex ν in K, let $\mathrm{Star}(\nu)$ denote the interior of the union of all simplices in K for which ν is a vertex. Then $\underline{U} = \{\mathrm{Star}(\nu) \mid \nu \text{ vertex}\}$ is an open cover of M. Given a finite collection of vertices ν_i, $0 \leq i \leq k$, the intersection of all $\mathrm{Star}(\nu_i)$ is empty unless every ν_i is a vertex of the same simplex. In that case, the intersection is exactly the interior of that simplex. Given a cochain $\sigma \in \mathrm{C}^n(\underline{U},\mathbb{Z}_M)$, we can define a simplicial cochain σ' as follows: if Δ is the simplex $\langle \nu_{i_0},\dots,\nu_{i_n}\rangle$, then we set $\sigma'(\Delta) = \sigma_{(i_0,\dots,i_n)}$, which gives us an isomorphism $\mathrm{C}^n(\underline{U},\mathbb{Z}_M) \to \mathrm{C}^n_{CW}(K,\mathbb{Z})$ of groups because

$$\mathbb{Z}_M\left(\bigcap_{j=0}^{n}\mathrm{Star}(\nu_j)\right) = \begin{cases}\mathbb{Z}, & \text{if the } \nu_j \text{ span an n-simplex,}\\ 0, & \text{otherwise.}\end{cases}$$

Moreover, this isomorphism clearly commutes with the coboundary operators so that we get an isomorphism of chain complexes and, hence, an isomorphism of cohomology $\mathrm{H}^*(\underline{U},\mathbb{Z}_M) \to \mathrm{H}^*_{\mathrm{CW}}(K,\mathbb{Z})$. By considering finer triangulations of K, the cover $\underline{U}$ may be taken arbitrarily small so that the isomorphism continues to hold after taking the direct limit. Thus,

$$\check{\mathrm{H}}^*(M,\mathbb{Z}_M) = \varinjlim \mathrm{H}^*(\underline{U},\mathbb{Z}_M) \cong \mathrm{H}^*_{\mathrm{CW}}(K,\mathbb{Z}) \cong \mathrm{H}^*_{\mathrm{sing}}(M,\mathbb{Z}).$$

□

Now let us turn to the second half of the proof of the de Rham theorem. Let Ω^k denote the sheaf of differential k-forms and Z^k the subsheaf of closed differential k-forms. In the following, d denotes the usual exterior differential of forms.

Lemma 1.11. $\mathrm{H}^*_{\mathrm{dR}}(M)$ *is isomorphic to* $\check{\mathrm{H}}^*(M,\mathbb{R}_M)$.

Proof. Consider the sequence

$$0 \to Z^k \hookrightarrow \Omega^k \xrightarrow{d} Z^{k+1} \to 0,$$

which is obviously exact at the first and second stage. Using lemma 1.4, we see that a sequence of sheaves is exact if and only if the induced sequence of stalks is exact at every point. The latter actually is an exact sequence. Indeed, the Poincaré lemma from differential topology states that locally every form is exact. Thus, the above sequence of sheaves is also exact at the third stage. Now let us consider the induced long exact sequence in cohomology,

$$\dots \to \check{\mathrm{H}}^{n-1}(M,Z^{k+1}) \to \check{\mathrm{H}}^n(M,Z^k) \to \check{\mathrm{H}}^n(M,\Omega^k) \to \check{\mathrm{H}}^n(M,Z^{k+1}) \to \check{\mathrm{H}}^{n+1}(M,Z^k) \to \dots$$

Clearly, Ω^k is a fine sheaf so that, by proposition 1.7, $\check{\mathrm{H}}^n(M,\Omega^k)$ vanishes for $n \geq 1$. Consequently, $\check{\mathrm{H}}^n(M,Z^{k+1})$ is isomorphic to $\check{\mathrm{H}}^{n+1}(M,Z^k)$ for $n \geq 1$. Iterating this statement yields

$$\check{\mathrm{H}}^1(M,Z^{n-1}) \cong \dots \cong \check{\mathrm{H}}^n(M,Z^0).$$

[2] The universal coefficients theorem also holds for cohomology of sheaves, see [1].

The sheaf Z^0 is the sheaf of closed zero forms, i.e. is exactly the sheaf $\mathbb{R}_M$. For the cohomology group on the left, we analyze the beginning of the long exact sequence,

$$\cdots \to \check{\mathrm{H}}^0(M,\Omega^k) \to \check{\mathrm{H}}^0(M,Z^{k+1}) \to \check{\mathrm{H}}^1(M,Z^k) \to \underbrace{\check{\mathrm{H}}^1(M,\Omega^k)}_{=0} \to \dots.$$

We invoke proposition 1.6 to check that $\check{\mathrm{H}}^0(M,\Omega^k)$ is exactly $\Omega^k(M)$ and $\check{\mathrm{H}}^0(M,Z^{k+1})$ is $Z^{k+1}(M)$. Combining the last two results gives rise to

$$\check{\mathrm{H}}^1(M,Z^{n-1}) \cong Z^n(M)/d\left(\Omega^{n-1}(M)\right) = \mathrm{H}^{\mathrm{n}}_{\mathrm{dR}}(M).$$

We are finished proving the lemma since now

$$\check{\mathrm{H}}^n(M,\mathbb{R}_M) = \check{\mathrm{H}}^n(M,Z^0) \cong \check{\mathrm{H}}^1(M,Z^{n-1}) \cong \mathrm{H}^{\mathrm{n}}_{\mathrm{dR}}(M).$$

□

Note that we can recast this result into a more general setting. The proof of the following theorem can be taken word for word from the lemma.

Theorem 1.12 (General de Rham Theorem). *Suppose we are given fine sheafs $\mathcal{G}^k$, subsheafs $\mathcal{F}^k$ of $\mathcal{G}^k$, and morphisms $d^k\colon \mathcal{G}^k \to \mathcal{F}^{k+1}$ such that the sequence*

$$0 \to \mathcal{F}^k \hookrightarrow \mathcal{G}^k \xrightarrow{d^k} \mathcal{F}^{k+1} \to 0$$

is exact for all $k \geq 0$. Then for all $n \geq 0$ we have

$$\check{\mathrm{H}}^n(M,\mathcal{F}^0) \cong \mathcal{F}^n(M)/d^{n-1}\left(\mathcal{G}^{n-1}(M)\right).$$

One instance where the above sequence is exact is for split differential forms on a complex manifold. Let Ω^p_h denote the sheaf of holomorphic p-forms.

Theorem 1.13 (Dolbeault). *If M is a complex manifold, then the Čech cohomology group $\check{\mathrm{H}}^q(M,\Omega^p_h)$ is isomorphic to the Dolbeault cohomology group $\mathrm{H}^{p,q}_{\bar{\partial}}(M)$.*

Proof. We take $\mathcal{G}^k$ to be sheaf of split differential forms of type (p,k) and $\mathcal{F}^k$ to be the subsheaf of $\bar{\partial}$-closed such forms. The analogue of the Poincaré lemma for the $\bar{\partial}$-operator[3] says that the short sequence in theorem 1.12 is exact. $\mathcal{F}^0$ is exactly Ω^p_h and the quotient on the right hand side of the conclusion of the general de Rham theorem is $\mathrm{H}^{p,q}_{\bar{\partial}}(M)$. □

[3] We prove the $\bar{\partial}$-Poincaré lemma independently in lemma 2.10.

2 Currents

2.1 Distributions and Currents

We will first deal with smooth functions and forms in euclidean space and later consider holomorphic equivalents in the complex setting. The C^p topology on the space $C_c^\infty(\mathbb{R}^n)$ of compactly supported smooth functions is defined by saying $\phi_n \to 0$ if $D^\alpha \phi_n \to 0$ uniformly for all multi-indices α with $|\alpha| \leq p$, where we use the notation[4] $D_j = \frac{\partial}{\partial x_j}$ and $D^\alpha = D_1^{\alpha_1} \dots D_n^{\alpha_n}$. This way, the C^p topology is finer than the C^q topology whenever $p > q$. We define the C^∞ topology to be the smallest topology containing all the C^p topologies. A *distribution* on $\mathbb{R}^n$ is a C^∞-continuous linear map $T\colon C_c^\infty(\mathbb{R}^n) \to \mathbb{C}$, and we denote the vector space of all distributions by $\mathcal{D}(\mathbb{R}^n)$. A distribution is said to be *of finite order* if it is C^p-continuous for some $p < \infty$. We introduce the notion of differentiation on the space of distributions by defnining

$$D_j T(\phi) = -T(D_j \phi).$$

The choice of minus sign will be motivated later. We will discuss some examples further below, but before that we want to introduce currents. Let $\Omega_c^q(\mathbb{R}^n)$ denote the space of smooth compactly supported q-forms on $\mathbb{R}^n$ with the topology induced from the C^∞ topology on $C_c^\infty(\mathbb{R}^n)$. The vector space of *currents of degree* q is the topological dual

$$\mathcal{C}^q(\mathbb{R}^n) = \left\{ T\colon \Omega_c^{n-q}(\mathbb{R}^n) \to \mathbb{C} \;\middle|\; T \text{ is linear and bounded} \right\}.$$

The exterior derivative for forms extends to currents via

$$d\colon \mathcal{C}^q(\mathbb{R}^n) \to \mathcal{C}^{q+1}(\mathbb{R}^n),\ (dT)(\phi) = (-1)^{q+1} T(d\phi)$$

such that we still have $d^2 = 0$. Let us briefly check that the new notions translate to a manifold setting. Let M be a smooth manifold. Clearly, all the definitions continue to work if we replace $\mathbb{R}^n$ by an open subset $U \subset \mathbb{R}^n$. Moreover, if $f\colon U \to V$ is a smooth diffeomorphism (for example, a transition map between two charts), then it induces a continuous isomorphism $C_c^\infty(V) \to C_c^\infty(U)$. Hence, we can can locally check whether a map $T\colon C_c^\infty(M) \to \mathbb{C}$ is continuous in the C^∞ topology, independent from the choice of chart. This argument shows that the analogous definition of $\mathcal{D}(M)$ is well-defined. The same holds for $\mathcal{C}^q(M)$.

Examples

Here, we want to review some of the standard examples. Let ψ be a locally integrable function on $\mathbb{R}^n$. We can associate to ψ a distribution T_ψ given by

$$T_\psi(\phi) = \int_{\mathbb{R}^n} \psi(x)\phi(x)dx.$$

This class of examples motivates the choice of minus sign in the definition of the differential. Indeed, if ψ is continuously differentiable, then we can compute by partial integration

$$(D_j T_\psi)(\phi) = -\int_{\mathbb{R}^n} \psi(x) \left(\frac{\partial}{\partial x_j} \phi(x) \right) dx = \int_{\mathbb{R}^n} \left(\frac{\partial}{\partial x_j} \psi(x) \right) \phi(x) dx = T_{D_j \psi}(\phi).$$

[4] We sometimes switch freely between the notations D, D_j, $\frac{\partial}{\partial x_j}$, ∂_j etc.

Another well-known distribution is the *dirac distribution*, or *δ-distribution*. δ simply evaluates its input at 0, i.e. $\delta(\phi) = \phi(0)$. For $n = 1$, the δ-distribution appears as the differential of a distribution of the first example: If ψ is the function on $\mathbb{R}$ that is 0 for $x < 0$ and is 1 for $x \geq 0$, then

$$(DT_\psi)(\phi) = -\int_{\mathbb{R}} \psi(x)\phi'(x)dx = -\int_0^\infty \phi'(x)dx = \phi(0).$$

We can consider similar examples for currents. Suppose $\psi = \sum_I \psi_I dx_I$ is a q-form on $\mathbb{R}^n$, whose coefficients ψ_I are locally integrable. Then the associated current $T_\psi \in \mathcal{C}^q(\mathbb{R}^n)$ is

$$T_\psi(\phi) = \int_{\mathbb{R}^n} \psi \wedge \phi.$$

As before, these examples motivate the choice of sign $(-1)^{q+1}$ in the definition of the differential, since we have

$$(dT_\psi)(\phi) = (-1)^{q+1}\int_{\mathbb{R}^n} \psi \wedge d\phi = \int_{\mathbb{R}^n} d\psi \wedge \phi - \int_{\mathbb{R}^n} d(\psi \wedge \phi) = T_{d\psi}(\phi),$$

for all smooth forms, where we used Stoke's theorem in the last equation. The second example that we will now introduce will be used later when we deal with cohomology theory. Suppose Γ is a piecewise smooth, oriented, $(n-q)$ chain in $\mathbb{R}^n$. Then Γ induces a current of degree q by integration,

$$T_\Gamma(\phi) = \int_\Gamma \phi.$$

Computing the differential gives a hint that this is suitable for cohomology theory later on,

$$(dT_\Gamma)(\phi) = (-1)^{q+1}\int_\Gamma d\phi = (-1)^{q+1}\int_{\partial\Gamma} \phi = (-1)^{q+1}T_{\partial\Gamma}(\phi),$$

where we used Stoke's theorem again.

Smoothening Distributions

We say that a distribution T is *smooth* if we have $T = T_\psi$ for some smooth function ψ on $\mathbb{R}^n$. We will show that any distribution can be approximated arbitrarily well by a smooth one. Pick a smooth, non-negative, radially symmetric, function $\chi\colon \mathbb{R}^n \to \mathbb{R}$ with a compact support containing the origin and such that $\int_{\mathbb{R}^n} \chi(x)dx = 1$. Define χ_ϵ to be the function $\frac{1}{\epsilon^n}\chi(\frac{x}{\epsilon})$. This function has the same properties as χ, but its support is $\epsilon{\cdot}\mathrm{supp}(\chi)$ Note that $T_{\chi_\epsilon} \to \delta$ as $\epsilon \to 0$ as distributions. Let $T \in \mathcal{D}(\mathbb{R}^n)$ be any distribution and denote by

$$\tilde{T}_\epsilon(x) = T_y(\chi_\epsilon(x-y))$$

the evaluation of T at the function $y \mapsto \chi_\epsilon(x-y)$. This is a continuous function $\mathbb{R}^n \to \mathbb{C}$ by the C^∞ continuity of T and is even smooth by linearity of T and smoothness of χ with derivatives given by

$$\frac{\partial}{\partial x_j}\tilde{T}_\epsilon(x) = \lim_{h\to 0}\frac{\tilde{T}_\epsilon(x+hx_j) - \tilde{T}_\epsilon(x)}{h} = \lim_{h\to 0} T_y\left(\frac{\chi_\epsilon(x+hx_j-y) - \chi_\epsilon(x-y)}{h}\right) = T_y\left(\frac{\partial}{\partial x_j}\chi_\epsilon(x-y)\right)$$

and, consequently, for any multi-index α

$$D^\alpha \tilde{T}_\epsilon(x) = T_y\left(D_x^\alpha \chi_\epsilon(x-y)\right).$$

Now let T_ϵ denote the smooth distribution $T_{\tilde{T}_\epsilon}$ associated to the function $\tilde{T}_\epsilon$. We need to check that $T_\epsilon \to T$ as $\epsilon \to 0$. This is the content of the next lemma. Given a function ϕ, we set

$$\phi_\epsilon(x) = \int_{\mathbb{R}^n} \phi(y)\chi_\epsilon(x-y)dy \left(= \int_{\mathbb{R}^n} \phi(y)\chi_\epsilon(y-x)dy\right).$$

Lemma 2.1. *The newly defined distribution satisfies:*
1. $(T_\psi)_\epsilon = T_{\psi_\epsilon}$, *for* $\psi \in C^\infty(\mathbb{R}^n)$,
2. $T_\epsilon(\phi) = T(\phi_\epsilon)$, *for* $\phi \in C_c^\infty(\mathbb{R}^n)$,
3. $(D^\alpha T)_\epsilon = D^\alpha(T_\epsilon)$.

Proof. The first part follows readily from the definition since $T_\psi(\chi_\epsilon(x-y)) = \psi_\epsilon(x)$. The second step uses the linearity of T,

$$T_\epsilon(\phi) = \int_{\mathbb{R}^n} \Big(T_y(\chi_\epsilon(x-y))\Big)\phi(x)dx = T_y\left(\int_{\mathbb{R}^n} \chi_\epsilon(x-y)\phi(x)dx\right) = T(\phi_\epsilon).$$

It suffices to prove the last part for an arbitrary $D = \frac{\partial}{\partial x_j}$. We first prove the assertion for a smooth distribution $T = T_\psi$. We calculate using integration by parts twice

$$\begin{aligned}
(DT_\psi)_\epsilon(\phi) &= -\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n} \psi(y)\frac{\partial}{\partial y_j}\chi_\epsilon(x-y)dy\right)\phi(x)dx \\
&= \int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n} \chi_\epsilon(x-y)\frac{\partial}{\partial y_j}\psi(y)dy\right)\phi(x)dx \\
&= -\int_{\mathbb{R}^n} \frac{\partial}{\partial x_j}\phi(x)\left(\int_{\mathbb{R}^n} \chi_\epsilon(y-x)\psi(y)dy\right)dx \\
&= T_{\psi_\epsilon}\left(-\frac{\partial}{\partial x_j}\phi\right) = (T_\psi)_\epsilon\left(-\frac{\partial}{\partial x_j}\phi\right) = D((T_\psi)_\epsilon)(\phi).
\end{aligned}$$

In particular, for smooth functions ψ we get $(D\psi)_\epsilon = D(\psi_\epsilon)$, i.e. not just as a distributional equation. For a general distribution T, we can conclude

$$(DT)_\epsilon(\phi) \overset{\text{part 2}}{=} (DT)(\phi_\epsilon) = -T\big(D(\phi_\epsilon)\big) = -T\big((D\phi)_\epsilon\big) \overset{\text{part 2}}{=} -T_\epsilon(D\phi) = D(T_\epsilon)(\phi).$$

$\square$

Currents as Differential Forms

Convention: we assume that all multi-indices are ordered. We would like to view a current $T \in \mathcal{C}^q(\mathbb{R}^n)$ as some sort of differential form. Firstly, define distributions $T_I \in \mathcal{D}(\mathbb{R}^n)$ for every multi-index I by

$$T_I(\phi) = T\left(\phi \star dx_I\right),$$

where $\star$ denotes the Hodge star operator. The only thing we need to know about the Hodge star operator is that $\star dx_I = \pm dx_{\star I}$, where we write $\star I$ for the multi-index containing every index that is not contained in I. The sign depends on whether "$I \cup \star I$" is an even or odd permutation of $(1, \dots, n)$. Let us symbolically write $dx_1 \wedge \dots \wedge dx_n$ for 1. In particular, we have

$$dx_I \wedge dx_J = \begin{cases} 1, & \text{if } dx_J = \star dx_I, \\ -1 & \text{if } dx_J = -\star dx_I, \\ 0, & \text{if } dx_J \neq \pm \star dx_I. \end{cases}$$

We would like to identify a current T with the "differential form" given by

$$\sum_{|I|=q} T_I dx_I,$$

which has distributions as coefficients. We interpret this expression as follows: given a differential form $\phi dx_J \in \Omega_c^{n-q}(\mathbb{R}^n)$, we set

$$T_I dx_I(\phi dx_J) = T_I(\phi) dx_I \wedge dx_J,$$

where $dx_I \wedge dx_J$ stands for 0, 1 or -1 as noted above. Then this identification works out for if ω is a form $\sum_{|J|=n-q} \phi_J dx_J \in \Omega_c^{n-q}(\mathbb{R}^n)$, then

$$\begin{aligned}\sum_{|I|=q} T_I dx_I(\omega) &= \sum_{|I|=q} \sum_{|J|=n-q} T_I(\phi_J) dx_I \wedge dx_J = \sum_{|J|=n-q} \operatorname{sign}(\star J \cup J) T_{\star J}(\phi_J) \\ &= \sum_{|J|=n-q} T(\phi_J \underbrace{\operatorname{sign}(\star J \cup J) \star dx_{\star J}}_{=dx_J}) = \sum_{|J|=n-q} T(\phi_J dx_J) = T(\omega).\end{aligned}$$

This is compatible with the usual differential of forms in light of

$$dT = (-1)^{q+1} \sum_{k=1}^{n} \sum_{|I|=q} (T_I \circ \partial_k)\, dx_I \wedge dx_k = \sum_{k=1}^{n} \sum_{|I|=q} (\partial_k T_I) dx_k \wedge dx_I.$$

If T is the current associated to a differential form

$$\psi = \sum_I h_I dx_I,$$

then the distributions T_I are (as expected) the distributions associated to h_I. The same point of view makes sense for currents of split type in the complex case, i.e. linear bounded maps from the space of compactly supported forms of split type (p,q) to $\mathbb{C}$. The operator

$$\bar\partial \colon \mathcal{C}^{p,q}(M) \to \mathcal{C}^{p,q+1}(M),\ (\bar\partial T)(\phi) = (-1)^{p+q+1} T(\bar\partial \phi)$$

whose square is zero, as well as $\partial \colon \mathcal{C}^{p,q}(M) \to \mathcal{C}^{p+1,q}(M)$, are also compatible with the point of view of differential forms.

Remark 2.2. *Caution is needed in the complex case: A (p,p)-current may take both ω and $\overline{\omega}$ as an input for $\omega \in \Omega_c^{n-p,n-p}(\mathbb{C}^n)$. However, when considering expressions of the form $\overline{T(\omega)}$, this really just conjugates the complex number $T(\omega)$ and not the differential form representing T. To illustrate what we mean, consider a (n,n)-current T. A (n,n)-current is just a distribution, so the representation as a form becomes $S dz_1 \wedge d\bar z_1 \wedge \cdots \wedge dz_n \wedge d\bar z_n$ for some distribution S. Then $\overline{T(\phi)}$, $\phi \in C_c^\infty(\mathbb{C}^n)$, is exactly $\overline{S(\phi)}$ and not*

$$\begin{aligned}\overline{S(\phi) dz_1 \wedge d\bar z_1 \wedge \cdots \wedge dz_n \wedge d\bar z_n} &= \overline{S(\phi)} d\bar z_1 \wedge dz_1 \wedge \cdots \wedge d\bar z_n \wedge dz_n \\ &= (-1)^n \overline{S(\phi)} dz_1 \wedge d\bar z_1 \wedge \cdots \wedge dz_n \wedge d\bar z_n.\end{aligned}$$

In particular, we may not apply the usual complex conjugation for forms to the form representation of a current.

Lastly, note that this point of view enables us to also smoothen currents simply by smoothing the distribution coefficients,

$$T_\epsilon = \sum_{|I|=q} (T_I)_\epsilon dx_I.$$

2.1.1 Regularity of the Differential Operator

Extend the *Laplacian* to distributions by

$$\Delta = -\sum_j \frac{\partial^2}{\partial x_j^2}.$$

A distribution is said to be *harmonic* if $\Delta T = 0$. If T is the distribution T_ψ associated to some function ψ, then T is a harmonic distribution if and only if ψ is a harmonic function. This even holds for all distributions as the next lemma shows.

Lemma 2.3. *Any harmonic distribution on $\mathbb{R}^n$ is the associated distribution of some harmonic function.*

Proof. Given a function $\psi \in C_c^\infty(\mathbb{R}^n)$, recall the notation

$$\psi_\epsilon(x) = \int_{\mathbb{R}^n} \psi(y)\chi_\epsilon(x-y)dy,$$

where χ_ϵ is the function from the chapter on the smoothing of distributions. For a special choice of such function χ, the integral defining ψ_ϵ becomes exactly the formula for the mean value property of harmonic functions. Thus, if ψ is harmonic, then $\psi_\epsilon = \psi$. Now let T_ϵ be the smoothing of T. In the remainder of the proof, we will use the properties of the smoothing established in lemma 2.1. Since $\Delta T_\epsilon = (\Delta T)_\epsilon = 0$, the smoothing T_ϵ is the distribution associated to a harmonic function ψ_ϵ. Then for the smoothing of the smoothing

$$(T_\epsilon)_\delta = (T_{\psi_\epsilon})_\delta = T_{(\psi_\epsilon)_\delta} = T_{\psi_\epsilon} = T_\epsilon.$$

In particular, if we evaluate T_ϵ on a test function $\phi \in C_c^\infty(\mathbb{R}^n)$, then

$$T_\epsilon(\phi) = (T_\epsilon)_\delta(\phi) = T_\epsilon(\phi_\delta).$$

Letting ϵ tend to 0, $T_\epsilon(\phi)$ becomes $T(\phi)$ and the right hand side becomes $T(\phi_\delta) = T_\delta(\phi)$. In other words, we conclude that $T = T_\delta$, which is a distribution associated to a harmonic function. $\square$

The analogue statement for the inhomogeneous equality is an immediate corollary from this lemma plus the classical solution to the Poisson formula.

Corollary 2.4. *If a distribution $T \in \mathcal{D}(\mathbb{R}^n)$ satisfies $\Delta T = T_\eta$ for some $\eta \in C_c^\infty(\mathbb{R}^n)$, then T is the distribution associated to some $\psi \in \mathbb{C}^\infty(\mathbb{R}^n)$ with $\Delta\psi = \eta$.*

Proof. It is a classical result that we can take a smooth function ρ with $\Delta\rho = \eta$. Such a solution can be constructed by using Green's formula. Then $\Delta(T - T_\rho) = 0$, so by the preceeding lemma $T - T_\rho = T_\psi$ for some harmonic function ψ. In particular, $T = T_{\psi+\rho}$. $\square$

The last two results were statements about the global case. The same works for the local case, too.

Lemma 2.5. *If $T \in \mathcal{D}(U)$ is a harmonic distribution on some open subset $U \subset \mathbb{R}^n$, then T is the associated distribution to some function that is harmonic inside U.*

Proof. The obstruction we face is that the smoothing ϕ_ϵ may have support outside U. However, fixing a relatively compact open subset $V \subset U$, we can pick ϵ small enough so that for any function $\phi \in C_c^\infty(V)$, the support of ϕ_ϵ is contained in U. Then the smoothing T_ϵ is defined on V and equals T_{ψ_V} for some harmonic function ψ_V on V. For any $\phi \in C_c^\infty(V)$ we get $T_{\psi_V}(\phi) = T_\epsilon(\phi) \equiv T(\phi)$. Doing the same on any relatively compact open set $V \subset W \subset U$, we obtain T_{ψ_W} with $\psi_V = \psi_W|_V$. This proves the lemma. $\square$

As a consequence, we obtain the analgoue statement for the $\bar{\partial}$-operator.

Theorem 2.6 (Regularity of the $\bar{\partial}$-operator)**.** *If a distribution $T \in \mathcal{D}(U)$, where U is an open subset of $\mathbb{C}^n$, is $\bar{\partial}$-closed, then T is the distribution associated to some holomorphic function on U.*

Proof. Note that $\bar{\partial}T = 0$ implies $\Delta T = 0$. Thus, the preceeding lemma yields a harmonic function f with $T = T_f$. But this function must be holmorphic since $0 = \bar{\partial}T = \bar{\partial}T_f = T_{\bar{\partial}f}$. □

This was a theorem for distributions and, indeed, we cannot simply replace the word "distribution" by "current". Fortunately, we can do so with an additional assumption on the degree of the current, which is sufficient for later applications.

Corollary 2.7 (Regularity of the $\bar{\partial}$-operator for currents)**.** *Suppose T is a current of type $(p, 0)$ on some open subset $U \subset \mathbb{C}^n$. If T is $\bar{\partial}$-closed, then it is a holomorphic differential form, i.e. a smooth differential form with holomorphic coefficients.*

Proof. Write the current as $T = \sum_{|I|=p} T_I dz_I$. Because its anit-antiholmorphic degree is 0, $\bar{\partial}T = 0$ reduces to $\bar{\partial}T_I = 0$ for every I. Applying the previous theorem yields the corollary. □

2.1.2 Cohomology of Currents

Since the differential for currents satisfies $d^2 = 0$, we get a complex of currents $(\mathcal{C}^*(M), d)$ and, hence, an associated cohomology $\mathrm{H}^*(\mathcal{C}, M)$. In the complex case, we also get a complex $(\mathcal{C}^{p,*}(M), \bar{\partial})$ and an associated cohomology $\mathrm{H}^{p,*}_{\bar{\partial}}(\mathcal{C}, M)$. We would like to show the following relation to well-known cohomology theories.

Theorem 2.8. *The cohomology of currents $\mathrm{H}^*(\mathcal{C}, M)$ is isomorphic to the de Rham cohomology $\mathrm{H}^*_{\mathrm{dR}}(M)$.*

Theorem 2.9. *If the manifold is complex, then the cohomology of complex currents $\mathrm{H}^{p,*}_{\bar{\partial}}(\mathcal{C}, M)$ is isomorphic to the Dolbeault cohomology $\mathrm{H}^{p,*}_{\bar{\partial}}(M)$.*

Just like the proof of the Dolbeault theorem was very similar to the proof of the de Rham theorem, it is the same for these two theorems. We will write down the proof for the second one only.

Proof of theorem 2.9. The idea is to use the general de Rham theorem again. There is an obvious fine sheaf $\mathcal{C}^{p,q}$ of currents of type (p, q). Let $\mathcal{F}^q$ denote the kernel sheaf of $\bar{\partial}\colon \mathcal{C}^{p,q} \to \mathcal{C}^{p,q+1}$. If we can show that the sequence

$$0 \to \mathcal{F}^q \hookrightarrow \mathcal{G}^q \xrightarrow{\bar{\partial}} \mathcal{F}^{q+1} \to 0$$

is exact for all $q \geq 0$, then by the general de Rham theorem 1.12

$$\check{\mathrm{H}}^q(M, \mathcal{F}^0) = \mathcal{F}^q(M)/\bar{\partial}\left(\mathcal{C}^{p,q-1}(M)\right) = \mathrm{H}^{p,q}_{\bar{\partial}}(\mathcal{C}, M).$$

The sheaf $\mathcal{F}^0$ is actually Ω^p_h by the regularity of the $\bar{\partial}$-operator, corollary 2.7. Thus, the check cohomology group on the left is exactly $\mathrm{H}^{p,q}_{\bar{\partial}}(M)$ by the Dolbeault theorem. The only stage of the sequence where being exact is not obvious is the third one. As before, by considering the sequence of the stalks, it suffices to show that locally every $\bar{\partial}$-closed current is exact. Hence, the proof of this theorem reduces to the $\bar{\partial}$-Poincaré lemma below. □

Lemma 2.10 ($\bar{\partial}$-Poincaré lemma for currents)**.** *A $\bar{\partial}$-closed current is locally $\bar{\partial}$-exact.*

Proof. Assume T is a $\bar{\partial}$-closed (p,q)-current and V is a small open set in $\mathbb{C}^n$. We will construct a $\bar{\partial}$-antiderivative $\tilde{T}$ of T on V. Define $T_c(\phi)$ to be $T(\rho\cdot\phi)$, where $\rho\colon \mathbb{C}^n \to [0,1]$ is a smooth function that is 1 on V and 0 outside some slightly larger $U(\rho)$. Then T_c agrees with T on V and is zero on $\mathbb{C}^n\setminus U(\rho)$, i.e. $T_c(\phi)=0$ for forms with support in $\mathbb{C}^n\setminus U(\rho)$. T_c is also $\bar{\partial}$-closed on V since $\bar{\partial}\rho$ has support in $U(\rho)\setminus V$,

$$\bar{\partial}T_c(\phi) = (-1)^{p+q+1}T(\rho\bar{\partial}\phi) = (-1)^{p+q+1}T(\bar{\partial}(\rho\phi)) + (-1)^{p+q}T(\underbrace{\bar{\partial}\rho\wedge\phi}_{=0}) = \bar{\partial}T(\rho\phi) = 0.$$

Next, we invoke proposition 2.11 from below, which states that there is a continuous linear operator

$$K\colon \Omega_c^{p,q}(\mathbb{C}^n) \to \Omega^{p,q-1}(\mathbb{C}^n)$$

with $\bar{\partial}\circ K - K\circ\bar{\partial} = (-1)^{p+q}\mathrm{id}$. With this, we can define a new $(p,q-1)$-current by $\tilde{T}(\phi) = T_c(K(\phi))$. Even though the image of ϕ under K may not have compact support, this is well-defined by the additional property of T_c. Then on V

$$\begin{aligned}\bar{\partial}\tilde{T}(\phi) &= (-1)^{p+q}T_c\Big(K\circ\bar{\partial}(\phi)\Big) = (-1)^{p+q}T_c\Big(\big(\bar{\partial}\circ K - (-1)^{p+q+1}\mathrm{id}\big)(\phi)\Big)\\ &= -\bar{\partial}T_c\Big(K(\phi)\Big) + T_c(\phi) = T(\phi) + (-1)^{p+q+1}T(\bar{\partial}\rho\wedge K(\phi)).\end{aligned}$$

By taking bump functions ρ_ϵ with $U(\rho_\epsilon)\setminus V \to \emptyset$, the remaining term vanishes in the limit. □

In the proof of the lemma, we referred to the following result.

Proposition 2.11. *There is a continuous linear operator*

$$K\colon \Omega_c^{p,q}(\mathbb{C}^n) \to \Omega^{p,q-1}(\mathbb{C}^n)$$

such that $\bar{\partial}\circ K - K\circ\bar{\partial} = (-1)^{p+q}\mathrm{id}$.

This is related to the so called Bochner-Martinelli kernel, which is the content of the next subchapter. In that subchapter, we will prove this proposition. However, we first want to explore the isomorphisms between the cohomology theories. The downside of the general de Rham theorem is that it is not constructive. Fortunately, we can still find the isomorphism. In fact, they are quite easy to write down. The map $\Omega^*(M)\to\mathcal{C}^*(M)$ that maps a form ψ to the associated current T_ψ descends to an injective homomorphism $\mathrm{H}^*_{\mathrm{dR}}(M)\to\mathrm{H}^*(\mathcal{C},M)$, which must therefore be an isomorphism. The isomorphism in the complex case is analogous. In the examples section we also discussed currents incuded by integration over a piecewise smooth oriented chain. Moreover, we calculated that $dT_\Gamma = (-1)^{q+1}T_{\partial\Gamma}$. Using that the singular cohomology and the piecewise smooth singular cohomology are isomorphic, we can therefore find a map $\mathrm{H}^*_{\mathrm{sing}}(M)\to\mathrm{H}^*(\mathcal{C},M)$ that sends $[\Gamma]$ to $[T_\Gamma]$. If $\partial\Gamma$ and $\partial\Gamma'$ are not the same, then we can find a form ϕ with $\int_{\partial\Gamma}\phi \neq \int_{\partial\Gamma'}\phi$. Thus, the above map also is an injective homomorphism. Since both $\mathrm{H}^*_{\mathrm{sing}}(M)$ and $\mathrm{H}^*(\mathcal{C},M)$ are isomorphic to the de Rham cohomology, this injective map also is an isomorphism.

2.1.3 The Bochner-Martinelli Kernel

In this section, we want to find the homotopy operator from proposition 2.11. This will be done by introducing the so called Bochner-Martinelli kernel. The construction relies on the fundamental solution to the Laplace equation. Since this solution needs to be expressed differently for the cases $n=1$ and $n\geq 2$ (once using the logarithm and once using a negative power), we will only consider the case $n\geq 2$.

For $n = 1$, the definitions have to be appropiately adapted but the arguments do not change. Begin by considering the smooth differential form on $\mathbb{C}^n \setminus \{0\}$ given by

$$\tilde{\mathrm{k}}_{\mathrm{BM}}(z) = \frac{c_n}{n-1} \sum_{j=1}^{n} (-1)^j \left(\frac{\partial}{\partial z_j} \frac{1}{|z|^{2n-2}} \right) d\bar{z}_1 \wedge \cdots \wedge \widehat{d\bar{z}_j} \wedge \cdots \wedge d\bar{z}_n \wedge dz_1 \wedge \cdots \wedge dz_n,$$

where c_n is a suitable complex contant specified later. Consequently,

$$d\tilde{\mathrm{k}}_{\mathrm{BM}}(z) = \bar{\partial}\tilde{\mathrm{k}}_{\mathrm{BM}}(z) = -\frac{c_n}{n-1} \left(\sum_{j=1}^{n} \frac{\partial}{\partial \bar{z}_j} \frac{\partial}{\partial z_j} \frac{1}{|z|^{2n-2}} \right) d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_n \wedge dz_1 \wedge \cdots \wedge dz_n.$$

Since $\tilde{\mathrm{k}}_{\mathrm{BM}}$ is locally integrable (even at 0), we can regard it as a current on $\mathbb{C}^n$. Recall that $|z|^{2-2n}$ is (up to a constant factor) the fundamental solution to the Laplace equation. Thus, if we choose the constant c_n accordingly, then the right hand side of the last equation viewed as a (n,n)-current is exactly the dirac distribution δ. Set

$$\tau \colon \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}^n, \ (z, \zeta) \mapsto \zeta - z$$

and define the *Bochner-Martinelli kernel* as $\mathrm{k}_{\mathrm{BM}} = \tau^* \tilde{\mathrm{k}}_{\mathrm{BM}}$. This is a smooth differential form on $\mathbb{C}^n \times \mathbb{C}^n$ minus the diagonal but as it inherits the local integrability of $\tilde{\mathrm{k}}_{\mathrm{BM}}$ we can also consider it as a current on $\mathbb{C}^n \times \mathbb{C}^n$. Let $[\Delta]$ denote the current given by $\tau^*\delta$. In other words, $[\Delta](\phi)$ is exactly the integral of ϕ along the diagonal in $\mathbb{C}^n \times \mathbb{C}^n$, for $\phi \in \Omega_c^0(\mathbb{C}^n \times \mathbb{C}^n) = C_c^\infty(\mathbb{C}^n \times \mathbb{C}^n)$. Then we get

$$d\mathrm{k}_{\mathrm{BM}} = \bar{\partial}\mathrm{k}_{\mathrm{BM}} = \tau^* \bar{\partial}\tilde{\mathrm{k}}_{\mathrm{BM}} = \tau^*\delta = [\Delta].$$

We are (almost) ready to construct the homotopy operator. The next result is the main step.

Proposition 2.12 (Bochner-Martinelli formula). *For every $\phi \in \Omega_c^q(\mathbb{C}^n)$ it holds that*

$$d \int_{\zeta \in \mathbb{C}^n} \phi(\zeta) \wedge \mathrm{k}_{\mathrm{BM}}(z,\zeta) - \int_{\zeta \in \mathbb{C}^n} d\phi(\zeta) \wedge \mathrm{k}_{\mathrm{BM}}(z,\zeta) = (-1)^q \phi(z),$$

where the first differential operator d is differentiation of currents.

Proof. Consider the current

$$T(z,\zeta) = \phi(\zeta) \wedge \mathrm{k}_{\mathrm{BM}}(z,\zeta) \wedge \psi(z),$$

where $\psi \in \Omega_c^{2n+1-q}(\mathbb{C}^n)$. Using $d\mathrm{k}_{\mathrm{BM}} = [\Delta]$, we can compute the differential of T to be

$$dT = d\phi \wedge \mathrm{k}_{\mathrm{BM}} \wedge \psi + (-1)^q \phi \wedge [\Delta] \wedge \psi + (-1)^{q+1} \phi \wedge \mathrm{k}_{\mathrm{BM}} \wedge d\psi.$$

For the smooting T_ϵ of T, we can use the regular Stokes' theorem,

$$0 = \int_{\mathbb{C}^n \times \mathbb{C}^n} d(T_\epsilon) = \int_{\mathbb{C}^n \times \mathbb{C}^n} (dT)_\epsilon \xrightarrow{\epsilon \to 0} \int_{\mathbb{C}^n \times \mathbb{C}^n} dT.$$

By definition of $[\Delta]$, the following identity of integrals is immediate,

$$\int_{\mathbb{C}^n \times \mathbb{C}^n} \phi(\zeta) \wedge [\Delta] \wedge \psi(z) = \int_{\mathbb{C}^n} \phi(z) \wedge \psi(z).$$

Combining the previous two statements yields

$$0 = \int_{\mathbb{C}^n \times \mathbb{C}^n} d\phi \wedge \mathrm{k}_{\mathrm{BM}} \wedge \psi + (-1)^q \int_{\mathbb{C}^n} \phi(z) \wedge \psi(z) + (-1)^{q+1} \int_{\mathbb{C}^n \times \mathbb{C}^n} \phi \wedge \mathrm{k}_{\mathrm{BM}} \wedge d\psi.$$

If S denotes the $(2n-1+q)$-current associated to $\phi \wedge \mathrm{k}_{\mathrm{BM}}$, then the right-most term is exactly $-dS(\psi)$. Thus, the above equality translates into the equation of currents

$$0 = \int_{\mathbb{C}^n \times \mathbb{C}^n} d\phi \wedge \mathrm{k}_{\mathrm{BM}} + (-1)^q \phi - dS,$$

which is what we wanted to show. □

Let ${\mathrm{k}_{\mathrm{BM}}}^{p,q}$ denote the (p,q)-type part with respect to z of k_{BM}. Note that the type of ${\mathrm{k}_{\mathrm{BM}}}^{p,q}$ with respect to ζ is then $(n-p, n-q-1)$. Given a smooth differential form $\phi \in \Omega_c^{p,q}(\mathbb{C}^n)$, define $K(\phi)$ by

$$K(\phi)(z) = \int_{\zeta \in \mathbb{C}^n} \phi(\zeta) \wedge {\mathrm{k}_{\mathrm{BM}}}^{p,q-1}(z, \zeta).$$

Let us show that this is a well-defined smooth differential form on $\mathbb{C}^n$.

Lemma 2.13. *K defines a linear operator $\Omega_c^{p,q}(\mathbb{C}^n) \to \Omega^{p,q-1}(\mathbb{C}^n)$. Furthermore, K is continuous.*

Proof. That $K(\phi)$ is a well-defined form follows from the fact that the singularity of k_{BM} along $\{\zeta = z\}$ is integrable. The fact that this is a smooth differential form follows from the exact definition of k_{BM}. The details of this argument are fundamental analytical observations and are omitted. For a reference, see [5, p. 64ff]. This establishes the first part of the lemma. To show that K is continuous, given an element $\phi \in \Omega_c^{p,q}(\mathbb{C}^n)$, define $d^{\alpha\bar{\beta}}\phi$ to be the differential form obtained by replacing the coefficients of ϕ by their $\partial^\alpha \bar{\partial}^\beta$-th derivative. It holds that $d^{\alpha\bar{\beta}} \circ K = K \circ d^{\alpha\bar{\beta}}$. Now suppose that $(\phi_k)_{k\geq 0} \subset \Omega_c^{p,q}(\mathbb{C}^n)$ converges to 0. This means that for every derivative $d^{\alpha\bar{\beta}}\phi_k$ the coefficients converge uniformly to 0. But then the same can be said of the coefficients of $d^{\alpha\bar{\beta}}K(\phi_k) = K(d^{\alpha\bar{\beta}}\phi_k)$. In other words, $K(\phi)$ converges to 0, which finishes the proof. □

Lastly, we need to show that K is exactly the required homotopy operator. We can consider all the (p,q)-types in the Bochner-Martinelli formula seperately to find that the equations of currents

$$\bar{\partial}K(\phi) - K(\bar{\partial}\phi) = (-1)^{p+q}\phi$$

hold. Because $K(\phi)$ actually is a smooth differential form, this equation also holds as an equation of differential forms. We have now proved proposition 2.11.

2.2 Positive Currents

2.2.1 Poincaré Lemmata

The analogue of the $\bar{\partial}$-Poincaré lemma for the ∂-operator is an immediate consequence, but since we never explicitly stated it, let us rectify this now.

Lemma 2.14 (∂-Poincaré lemma)**.** *A ∂-closed current is locally ∂-exact.*

Proof. Suppose $\partial T = 0$ for some (p,q)-current T. Define a (q,p)-current by $T'(\phi) = T(\overline{\phi})$. Then T' is $\bar{\partial}$-closed,

$$\bar{\partial}T'(\phi) = (-1)^{q+p+1}T'(\bar{\partial}\phi) = (-1)^{p+q+1}T(\partial\overline{\phi}) = \partial T(\overline{\phi}) = 0.$$

By the $\bar{\partial}$-Poincaré lemma for currents, locally $T' = \bar{\partial}S'$ for some $(q, p-1)$-current S'. As before, set $S(\phi) = S'(\overline{\phi})$. Then

$$\partial S(\phi) = (-1)^{q+p} S(\partial\phi) = (-1)^{p+q} S'(\overline{\partial\phi}) = \bar{\partial}S'(\overline{\phi}) = T'(\overline{\phi}) = T(\phi).$$

This concludes the lemma. □

We can extend this to prove another analogue of a Poincaré lemma; this time for the composition $\partial\bar{\partial}$. The proof is not as trivial, but it still merely is a repeated application of the various Poincaré lemmata we know so far. However, let us first introduce some new relevant notions to cast this result in a prettier language. If a current has the same degree in both types, i.e. $T \in \mathcal{C}^{p,p}(M)$, then it can take both ϕ and $\overline{\phi}$ as input. We say that T is a *real current* if $\overline{T(\phi)} = T(\overline{\phi})$ for every $\phi \in \Omega_c^{n-p,n-p}(M)$. Caution is appropriate: a real current in this sense can still take complex values since the condition is not $\overline{T(\phi)} = T(\phi)$. We can analyze this definition by considering currents as forms. Suppose T is a (p, p)-current with representation as a differential form locally given by

$$T = \sum_{|I|=p=|J|} T_{I,J} dz_I \wedge d\bar{z}_J,$$

with distribution coefficients $T_{I,J}$. Evaluating it on the form

$$\omega = \sum_{|I|=p=|J|} \phi_{I,J} dz_{\star I} \wedge d\bar{z}_{\star J}$$

as well as on $\overline{\omega}$ yields the horrible looking formulas

$$T(\omega) = (-1)^{p(n-p)+\frac{n(n-1)}{2}} \sum_{|I|=p=|J|} \operatorname{sign}(I)\operatorname{sign}(J) T_{I,J}(\phi_{I,J}),$$
$$T(\overline{\omega}) = (-1)^{n(n-p)+\frac{n(n-1)}{2}} \sum_{|I|=p=|J|} \operatorname{sign}(I)\operatorname{sign}(J) T_{I,J}(\overline{\phi_{J,I}}).$$

Thus, $\overline{T(\omega)} = T(\overline{\omega})$ if $\overline{T_{I,J}(\phi_{I,J})} = (-1)^{n-p} T_{J,I}(\overline{\phi_{I,J}})$ for all I, J. More generally, T is real if and only $\overline{T_{I,J}(\phi)} = (-1)^{n-p} T_{J,I}(\overline{\phi})$ for any input $\phi \in C_c^\infty(\mathbb{C}^n)$ and all I, J. To verify that this definition is sensible, let us check compatibility with smooth forms. Suppose the coefficients $T_{I,J}$ are associated distributions $T_{h_{I,J}}$ for some smooth functions $h_{I,J}$. Then

$$\overline{T_{I,J}(\phi)} = \int_{\mathbb{C}^n} \overline{h_{I,J}\phi} d\bar{z}_1 \wedge dz_1 \wedge \cdots \wedge d\bar{z}_n \wedge dz_n = (-1)^n \int_{\mathbb{C}^n} \overline{h_{I,J}\phi} dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_n$$

so that T is a real current if and only if $\overline{h_{I,J}} = (-1)^p h_{J,I}$ for all I, J. It is common to write smooth (p, p)-forms with a complex factor in front,

$$\psi = i^p \sum_{|I|=p=|J|} h_{I,J} dz_I \wedge d\bar{z}_J,$$

because now T_ψ is a real current if and only if $\overline{h_{I,J}} = h_{J,I}$ for all I, J or, equivalently, $\overline{\psi} = \psi$. This is the usual condition we know from complex geometry for a form to be real. We would also like to introduce the notion of a positive current. Note that for a real (p, p)-current T and a $(n-p, 0)$-test form η we have

$$\overline{i^{n-p} T(\eta \wedge \overline{\eta})} = i^{n-p} T(\eta \wedge \overline{\eta}) \in \mathbb{R}.$$

Instead of stating the definition, let us deduce what it needs to look like in order to be compatible with the following requirement: we want integration over $\mathbb{C}^{n-p} \subset \mathbb{C}^n$ to define a positive current. Suppose $\eta = \sum_{|I|=p} \phi_I dz_{\star I}$ and abbreviate $\phi = \phi_{(n-p+1,\dots,n)}$. Then the current by integration over $\mathbb{C}^{n-p} \times \{0\}$ is

$$\begin{aligned} T(\eta \wedge \bar{\eta}) &= \int_{\mathbb{C}^{n-p}\times\{0\}} \phi\bar{\phi}\; dz_1 \wedge \cdots \wedge dz_{n-p} \wedge d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_{n-p} \\ &= (-1)^{\frac{(n-p)(n-p-1)}{2}} \int_{\mathbb{C}^{n-p}\times\{0\}} |\phi|^2\; dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge dz_{n-p} \wedge d\bar{z}_{n-p} \\ &= (-1)^{\frac{(n-p)(n-p-1)}{2}} (-2i)^{n-p} \int_{\mathbb{C}^{n-p}\times\{0\}} |\phi|^2\; dx_1 \wedge dy_1 \wedge \cdots \wedge dx_{n-p} \wedge dy_{n-p}. \end{aligned}$$

Thus, let us define a real (p,p)-current T to be *positive* if for any $\eta \in \Omega_c^{n-p,0}(\mathbb{C}^n)$

$$(-1)^{\frac{(n-p)(n-p-1)}{2}} i^{n-p} T(\eta \wedge \bar{\eta}) \geq 0.$$

As before, let us check compatibility with smooth forms. A smooth $(1,1)$-form

$$\psi = i \sum_{j,k=1}^{n} h_{j,k} dz_j \wedge d\bar{z}_k$$

is positive if the matrix with entries $h_{j,k}$ is positive definite at every point. The above formula with $T = T_\psi$, $p = 1$, and $\omega = \eta \wedge \bar{\eta}$ reads

$$T_\psi(\eta \wedge \bar{\eta}) = (-1)^{n-1+\frac{n(n-1)}{2}} i \sum_{j,k=1}^{n} \operatorname{sign}(j)\operatorname{sign}(k) T_{h_{j,k}}(\phi_j \bar{\phi}_k),$$

where $\operatorname{sign}(j)$ is just $(-1)^{j-1}$. Thus,

$$\begin{aligned} (-1)^{\frac{(n-1)(n-2)}{2}} i^{n-1} T_\psi(\eta \wedge \bar{\eta}) &= i^n \sum_{j,k=1}^{n} (-1)^{j+k} T_{h_{j,k}}(\phi_j \bar{\phi}_k) \\ &= i^n \int_{\mathbb{C}^n} \left(\sum_{j,k=1}^{n} (-1)^{j+k} \phi_j h_{j,k} \bar{\phi}_k \right) dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_n \\ &= 2^n \int_{\mathbb{C}^n} \left(\sum_{j,k=1}^{n} (-1)^{j+k} \phi_j h_{j,k} \bar{\phi}_k \right) dx_1 \wedge dy_1 \wedge \cdots \wedge dx_n \wedge dy_n. \end{aligned}$$

This term is positive for any choice of functions ϕ_j if and only if the matrix with entries $(-1)^{j+k} h_{j,k}$ is positive definite. This latter matrix is positive definite if and only if the matrix with entries $h_{j,k}$ is positive definite. Thus, positivity of currents is (almost) compatible with positivity of forms. We say "almost" because the definition of a positive current allows $T(\eta \wedge \bar{\eta}) = 0$. Hence, T_ψ is positive as a current if and only if the coefficient matrix $(h_{j,k})_{j,k}$ is positive semi-definite. We may say that a real (p,p)-current T is *strictly positive* if for any non-trivial $\eta \in \Omega_c^{n-p,0}(\mathbb{C}^n)$

$$(-1)^{\frac{(n-p)(n-p-1)}{2}} i^{n-p} T(\eta \wedge \bar{\eta}) > 0.$$

Then strictly positive smooth currents correspond exactly to positive smooth forms. However, for convenience, we will slightly distance ourselves from the usual convention and also call a smooth form positive if its associated current is positive (instead of strictly positive). Here is one last definition. A real locally integrable function ρ is said to be *plurisubharmonic* if the current $i\partial\bar{\partial}\rho$ is a positive current. Here, $\partial\bar{\partial}\rho$ is differentation of ρ as a distribution. The $\partial\bar{\partial}$-Poincaré lemma shows how every closed positive $(1,1)$-current is realized as such $i\partial\bar{\partial}\rho$.

Proposition 2.15 ($\partial\bar{\partial}$-Poincaré lemma for currents)**.** *A closed positive* $(1,1)$*-current* T *is locally of the form*

$$T = i\partial\bar{\partial}\rho$$

for some real (necessarily plurisubharmonic) function ρ*, which is unique up to adding the real part of a holomorphic function.*

Proof. Note that T being closed implies being both ∂- and $\bar{\partial}$-closed. This follows from the assertion that T is a real current. By the $\bar{\partial}$-Poincaré lemma for currents, T is locally of the form $-i\bar{\partial}S$ for some current S, which must be of type $(1,0)$. Then $\bar{\partial}(\partial S) = -\partial\bar{\partial}S = -i\partial T = 0$, so the regularity result for the $\bar{\partial}$-operator for currents implies that ∂S is a holomorphic differential form. By the standard Poincaré lemma, ∂S equals $d\omega$ for some holomorphic 1-form ω. Define a new current $S' = S - \omega$, which satisfies $T = -i\bar{\partial}S'$ because $\bar{\partial}\omega = 0$. Since also $\partial S' = 0$, the ∂-Poincaré lemma for currents gives us a $(0,0)$-current R with $S' = \partial R$ locally. Note that, while (n,n)-currents are exactly distributions by definition, we can also regard $(0,0)$-currents as distributions since their input is exactly differential forms of top degree. Hence, $\rho = (R + R')/2 = \mathrm{Re}(R(\phi))$, where R' is defined by $R'(\phi) = \overline{R(\overline{\phi})}$, is a real distribution. It follows from all the different stages in the proof that

$$\begin{aligned} \partial\bar{\partial}R &= -\bar{\partial}\partial R = -\bar{\partial}S' = -iT, \\ \partial\bar{\partial}R'(\phi) = -R'(\bar{\partial}\partial\phi) = -\overline{R(\partial\bar{\partial}\overline{\phi})} &= \overline{\bar{\partial}\partial R(\overline{\phi})} = \overline{iT(\overline{\phi})} = -iT(\phi). \end{aligned}$$

This proves that $i\partial\bar{\partial}\rho = T$. One can show that ρ actually is the distribution associated to a locally integrable function. It remains to verify the uniqueness property. Suppose σ is another solution, $T = i\partial\bar{\partial}\sigma$. Note that $\partial\bar{\partial}(\sigma - \rho) = 0$ implies $\Delta(\sigma - \rho) = 0$. Thus, $\sigma - \rho$ is a harmonic function. However, for a harmonic function it is well known that it is exactly the real part of a holomorphic function. ☐

We call such a function ρ a *potential of* T. If we replace local integrability by smoothness, then a plurisubharmonic function gives rise to a Kähler form. Conversely, the $\partial\bar{\partial}$-Poincaré lemma states that any Kähler form admits a local smooth potential since smoothness is preserved throughout the proof.

2.2.2 The Poincaré Lelong equation

Let us now turn to a specific class of currents. Suppose M is a complex manifold and $Z \subset M$ an analytic hypersurface. Denote by Z_* the set of regular points in Z. Then Z induces a current T_Z by integration over Z_*.

Lemma 2.16. *Let* p *denote the codimension of* Z *in* M*. Then*

$$T_Z \colon \Omega_c^{n-p,n-p}(M) \to \mathbb{C},\ \phi \mapsto \int_{Z_*} \phi$$

is a closed positive (p,p)*-current.*

Proof. Let us begin by showing that it is closed. We will show that even locally the integral of a closed form is always 0. Given a point in Z^*, take local coordinates $z = (z_1, \dots, z_n)$ in some neighborhood U

such that the projection π sending z to the first $n-p$ coordinates is a branched covering $Z^* \cap U \to \pi(U)$, branched over some hypersurface $D \subset \pi(U)$. Denote by D_ϵ a small ϵ-neighborhood of D in $\pi(U)$ and set

$$Z_\epsilon^* = (Z^* \cap U) \setminus \pi^{-1}(D_\epsilon).$$

Then by Stoke's theorem,

$$\int_{Z^* \cap U} d\phi = \lim_{\epsilon \to 0} \int_{Z_\epsilon^*} d\phi = \lim_{\epsilon \to 0} \int_{\partial \pi^{-1}(D_\epsilon)} \phi.$$

If we can show that the volume of ∂D_ϵ tends to 0 as ϵ tends to 0, then the volume of $\partial\pi^{-1}(D_\epsilon)$ also tends to 0 and we can conclude that T_Z is a closed current. To this end, let D_s denote the set of singular points of D and $D_1 = D \setminus D_s$. Similarly, define $D_2 = D_1 \setminus (D_1)_s$ and so on. Then each D_j is a submanifold of strictly smaller (real) dimension. In particular, the boundary of an ϵ-neighborhood D_ϵ^j will have volume approaching 0 as we shrink ϵ. Since ∂D_ϵ is contained in the countable union of the ∂D_ϵ^j, we are finished proving closedness. Finally, we need to prove positivity of T_Z. It suffices to work in a small open set since all input forms are compactly supported. Take coordinates such that Z_* is the zero set $\{z_{n-p+1} = \cdots = z_n = 0\}$. Then, locally, the current is just integration over $\mathbb{C}^{n-p} \times \{0\}$, for which we already established positivity. □

In order to find a potential of such a current, we first need a variation of the Cauchy integral formula.

Lemma 2.17. *Suppose p is a polynomial in one variable with roots $\{w_\nu\}_\nu$. Then we have the distributional equation*

$$\bar{\partial}\left(\frac{1}{2\pi i}\frac{p'(w)}{p(w)}dw\right) = \sum_\nu \delta_{w_\nu}.$$

Proof. We begin by proving the lemma for $p(w) = w - z$. Take a test function $\phi \in C_c^\infty(\mathbb{C})$. Away from $z \in \mathbb{C}$ it holds that

$$d\left(\frac{\phi}{w-z}dw\right) = \frac{\partial\phi}{\partial\bar{w}}\frac{1}{w-z}d\bar{w} \wedge dw$$

so that, by Stoke's theorem, for any disk D around z and any smaller disk $D_\epsilon \subset D$ of radius ϵ

$$\int_{\partial D_\epsilon} \frac{\phi}{w-z}dw = \int_{\partial D} \frac{\phi}{w-z}dw - \int_{D \setminus D_\epsilon} \frac{\partial\phi}{\partial\bar{w}}\frac{1}{w-z}d\bar{w} \wedge dw.$$

The integral in the middle vanishes if we take D so large that its boundary does not intersect the support of ϕ. The integral on the left becomes in polar coordinates

$$\int_{\partial D_\epsilon} \frac{\phi}{w-z}dw = \int_0^{2\pi} \phi(z + \epsilon e^{i\theta}) i d\theta \xrightarrow{\epsilon \to 0} 2\pi i \phi(z).$$

Next, we use $dw \wedge d\bar{w} = -2irdr \wedge d\theta$ and boundedness of $\frac{\partial\phi(w)}{\partial\bar{w}}$ by some constant c to conclude

$$\left|\int_{D_\epsilon} \frac{\partial\phi}{\partial\bar{w}}\frac{1}{w-z}dw \wedge d\bar{w}\right| \leq 2c\int_{D_\epsilon} |dr \wedge d\theta| \xrightarrow{\epsilon \to 0} 0.$$

This proves the special case

$$2\pi i\phi(z) = -\int_D \frac{\partial\phi}{\partial\bar{w}}\frac{1}{w-z}d\bar{w} \wedge dw = \bar{\partial}\left(\frac{1}{w-z}dw\right)(\phi).$$

We can write a general polynomial $p(w)$ as the product of the terms $(w - w_\nu)$ to find

$$\bar{\partial}\left(\frac{1}{2\pi i}\frac{p'(w)}{p(w)}dw\right) = \sum_\nu \bar{\partial}\left(\frac{1}{2\pi i}\frac{1}{w - w_\nu}dw\right) = \sum_\nu \delta_{w_\nu}.$$

□

Proposition 2.18 (Poincaré Lelong equation)**.** *Suppose a holomorphic function f on a complex manifold has divisor the analytic hypersurface Z. Then, as currents,*

$$T_Z = \frac{i}{\pi}\partial\bar{\partial}\log|f|.$$

Proof. Given $p \in Z$, we can pick local coordinates z such that $f(z)$ becomes the product of a Weierstrass polynomial $g(z)$ in z_n and a non-vanishing holomorphic function $h(z)$. This is possible by the Weierstrass Preparation Theorem, see [3, p. 8]. Since h is holomorphic and non-vanishing, we get

$$\partial\bar{\partial}\log|f| = \partial\bar{\partial}\log|g| + \frac{1}{2}\left(\partial\bar{\partial}\log h + \partial\bar{\partial}\log\bar{h}\right) = \partial\bar{\partial}\log|g|.$$

Thus, we may neglect h and assume $f = g$. Take a test function ϕ. Since f remains a Weierstrass polynomial when we rescale the z_n coordinate by a linear combination of the other coordinates ($z_n' = \gamma_1 z_1 + \cdots + \gamma_{n-1}z_{n-1} + z_n$), we may assume that the test function is of the form

$$\phi(z) = \alpha(z)dz_1 \wedge \cdots \wedge dz_{n-1} \wedge d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_{n-1}.$$

By renaming $w = z_n$ and $z = (z_1, \ldots, z_{n-1})$ we can rewrite this nicely as

$$\begin{aligned}\phi(z, w) &= \alpha(z, w)dz \wedge d\bar{z},\\ f(z, w) &= w^n + a_{n-1}(z)w^{n-1} + \cdots + a_0(z).\end{aligned}$$

We will write f' for the w-derivative of f. We can use Stoke's theorem on the w coordinate,

$$\begin{aligned}d_w\left(\log|f|^2\frac{\partial\alpha}{\partial\bar{w}}d\bar{w}\right) &= \left(\frac{f'}{f}dw + \frac{\overline{f'}}{\bar{f}}d\bar{w}\right) \wedge \left(\frac{\partial\alpha}{\partial\bar{w}}d\bar{w}\right) + \log|f|^2\frac{\partial^2\alpha}{\partial w\partial\bar{w}}dw \wedge d\bar{w}\\ &= \frac{f'}{f}\frac{\partial\alpha}{\partial\bar{w}}dw \wedge d\bar{w} + 2\log|f|\frac{\partial^2\alpha}{\partial w\partial\bar{w}}dw \wedge d\bar{w},\end{aligned}$$

and that ϕ is compactly supported so that the boundary term in Stoke's formula vanishes to find

$$\begin{aligned}\left(\frac{i}{\pi}\partial\bar{\partial}\log|f|\right)(\phi) &= -\frac{i}{\pi}\int\int\log|f|\frac{\partial^2\alpha}{\partial w\partial\bar{w}}d\bar{w} \wedge dw \wedge dz \wedge d\bar{z}\\ &= \frac{i}{2\pi}\int\left(\int\frac{f'}{f}\frac{\partial\alpha}{\partial\bar{w}}d\bar{w} \wedge dw\right)dz \wedge d\bar{z}.\end{aligned}$$

We can evaluate the inner integral with the previous lemma to deduce

$$\left(\frac{i}{\pi}\partial\bar{\partial}\log|f|\right)(\phi) = \int\sum_\nu \alpha(z, w_\nu(z))dz \wedge d\bar{z},$$

where $w_\nu(z)$ are the zeros of the polynomial $f(z, \cdot)$. As the zero set of f is exactly Z by hypothesis, this last integral is integration of ϕ over Z. □

2.2.3 The Lelong Number

Also due to Lelong is the Lelong number, which we construct next. Let $B(r)$ denote the ball of radius r around the origin in $\mathbb{C}^n$ and $B(r,R)$ the annulus with inner radius r and outer radius R. Set

$$\omega = \frac{i}{2}\partial\bar{\partial}||z||^2 = i\sum_{j=1}^{n} dz_j \wedge d\bar{z}_j,$$
$$\Omega = \frac{i}{2\pi}\partial\bar{\partial}\log||z||^2.$$

Given a smooth (p,p)-current T_ψ, define

$$\Theta(T_\psi, r) = \frac{1}{r^{2(n-p)}}\int_{B(r)} \psi \wedge \omega^{n-p}.$$

For an arbitrary (p,p)-current T, use the smoothing to extend Θ to T,

$$\Theta(T,r) = \limsup_{\epsilon\to 0}\Theta(T_\epsilon, r).$$

If T is a positive current, then $\Theta(T,r)$ is always a non-negative real number. Indeed, by construction, the smoothing of a positive current remains positive. Therefore, we only need to verify $\Theta(T_\psi, r) \geq 0$ for positive smooth forms ψ. This is the special case $\Psi = \omega$ in the lemma below.

Lemma 2.19. *If ψ is a positive smooth (p,p)-form and Ψ a positive smooth $(1,1)$-form, then the integral of $\psi \wedge \Psi^{n-p}$ over any domain is always a non-negative real number.*

Proof. Such an integral is certainly a real number as both ψ and Ψ are real forms. It suffices to show that $\psi \wedge \Psi$ is positive, because then we iteratively conclude that $\psi \wedge \Psi^{n-p}$ is positive and the definition of positivity for $p = n$ reads $T_{\psi\wedge\Psi^{n-p}}(|\phi|^2) \geq 0$ for any compactly supported smooth function ϕ. As Ψ is positive, its coefficient matrix is diagonalizable with only non-negative eigenvalues and we may perform a change of coordinates in which Ψ takes the form

$$\Psi = i\sum_{j=1}^{n} \alpha_j dw_j \wedge d\bar{w}_j$$

with functions α_j taking values in $\mathbb{R}_{\geq 0}$. Take a $(n-p-1,0)$-test form η and set $\eta_j = \sqrt{\alpha_j}dw_j \wedge \eta$. We simply calculate

$$T_{\psi\wedge\Psi}(\eta\wedge\bar{\eta}) = (-1)^{n-p-1} i\sum_{j=1}^{n} T_\psi(\eta_j \wedge \bar{\eta}_j)$$

and, subsequently, by positivity of ψ

$$(-1)^{\frac{(n-p-1)(n-p-2)}{2}} i^{n-p-1} T_{\psi\wedge\Psi}(\eta\wedge\bar{\eta}) = \sum_{j=1}^{n}(-1)^{\frac{(n-p-1)(n-p)}{2}} i^{n-p} T_\psi(\eta_j\wedge\bar{\eta}_j) \geq 0.$$

□

We may wish to take the limit to get rid of the r-dependence of our newly defined non-negative real number $\Theta(T,r)$. The next lemma enables us to do so.

Lemma 2.20. *Suppose T is a closed positive (p,p)-current. Then $\Theta(T,r)$ decreases as r decreases.*

Proof. It suffices to prove the lemma for smooth currents. Assume $T = T_\psi$. ψ being closed implies

$$\left(\frac{i}{2}\right)^{n-p} d\left(\psi \wedge \bar{\partial}||z||^2 \wedge \left(\partial\bar{\partial}||z||^2\right)^{n-p-1}\right) = \psi \wedge \omega^{n-p}.$$

Thus, by Stoke's theorem,

$$\Theta(T_\psi, r) = \frac{1}{r^{2(n-p)}} \left(\frac{i}{2}\right)^{n-p} \int_{\partial B(r)} \psi \wedge \bar{\partial}||z||^2 \wedge \left(\partial\bar{\partial}||z||^2\right)^{n-p-1}.$$

Another easy computation allows us to substitute a logarithm into the integral,

$$\bar{\partial} \log ||z||^2 = \frac{1}{||z||^2}\bar{\partial}||z||^2 \overset{z\in\partial B(r)}{=} \frac{1}{r^2}\bar{\partial}||z||^2$$

and, hence,

$$\bar{\partial} \log ||z||^2 \wedge \left(\partial\bar{\partial} \log ||z||^2\right)^{n-p-1} = \frac{1}{r^{2(n-p)}} \bar{\partial}||z||^2 \wedge \left(\partial\bar{\partial}||z||^2\right)^{n-p-1}$$

Lastly, since

$$\left(\frac{i}{2}\right)^{n-p} d\left(\bar{\partial} \log ||z||^2 \wedge \left(\partial\bar{\partial} \log ||z||^2\right)^{n-p-1}\right) = \pi^{n-p}\Omega^{n-p},$$

we can use Stoke's theorem again to find for $r < R$

$$\begin{aligned}\Theta(T_\psi, R) - \Theta(T_\psi, r) &= \left(\frac{i}{2}\right)^{n-p} \int_{\partial B(r,R)} \psi \wedge \bar{\partial} \log ||z||^2 \wedge \left(\partial\bar{\partial} \log ||z||^2\right)^{n-p-1} \\ &= \pi^{n-p} \int_{B(r,R)} \psi \wedge \Omega^{n-p}.\end{aligned}$$

Since T_ψ is a positive current, this integral must be non-negative by lemma 2.19. Indeed, Ω is positive as its coefficient matrix has entries

$$h_{j,k} = \frac{1}{||z||^2}\delta_{j,k} - \frac{\bar{z}_j z_k}{||z||^4}$$

and, therefore, has eigenvalues 0 (with multiplicity 1) and $1/||z||^2$ (with multiplicity $n-1$). $\square$

We can now define the *Lelong number* of a closed positive current $T \in \mathcal{C}^{p,p}(\mathbb{C}^n)$ as

$$\Theta(T) = \frac{1}{\pi^{n-p}} \lim_{r\to 0} \Theta(T, r).$$

Clearly, if $p \geq 1$ and $T = T_\psi$ is a smooth current, then

$$\Theta(T) \leq \text{const} \cdot \lim_{r\to 0} \frac{1}{r^{2(n-p)}} \text{vol}(B(r)) = 0.$$

For currents on a complex manifold, we can define a point-wise Lelong number $\Theta(T, p)$ by taking a chart centered at p.

2.3 Application: Intersection of Analytic Subvarieties

Currents can also be used to compute intersection numbers. Suppose M is an oriented, compact, complex manifold and V and W are two analytic subvarieties of dimensions p and $n-p$, respectively. The intersection theory can be done in the cohomology of currents simply by "pulling it back" from the Dolbeault cohomology via the isomorphism from theorem 2.9. We will prove the following result:

Theorem 2.21. *Suppose V and W intersect in only finitely many points. The intersection number of V and W inside M is exactly the sum of the intersection multiplicities of each intersection point,*

$$V \cdot W = \sum_{p \in V \cap W} m_p(V, W).$$

Proof. Suppose first that W is smooth. We may assume that $V \cap W$ consists of a single point p_0. Take coordinates $(z, w) = (z_1, \ldots, z_p, w_1, \ldots, w_{n-p})$ in a neighborhood U of p_0 such that W is given by $\{z = 0\}$ and the projection $(z, w) \mapsto z$ is a finite cover. Let S be the $(p, p-1)$-current on U given by the variation of the Bochner-Martinelli form (variation in the sense that S inherits trivial w-coordinate dependence)

$$S = \text{const} \cdot \sum_{j=1}^{p} (-1)^j \left(\frac{\partial}{\partial z_j} \frac{1}{|z|^{2n-2}} \right) d\bar{z}_1 \wedge \ldots \widehat{d\bar{z}_j} \wedge \cdots \wedge d\bar{z}_p \wedge dz_1 \wedge \cdots \wedge dz_p.$$

Then $dS = \bar{\partial} S$ is integration over $\{z = 0\}$, i.e. equals $T_W|_U$, where T_W is the current associated to W by integration. Let U' denote a smaller neighborhood. Pick some bump function ρ that is 1 on U' with support in U. Then the current

$$T' = T_W - d(\rho S)$$

vanishes in U' and, hence, is smooth on V. Since it lies in the same cohomology class as T_W, we recover the intersection number (this follows from the analogue result we know for Dolbeault cohomology and intersection numbers)

$$V \cdot W = \int_V T'.$$

Abbreviate $V_\epsilon = V \cap U_\epsilon$, where U_ϵ is $\{||z|| < \epsilon,\ ||w|| < \epsilon\} \subset U$. Using that T' vanishes in U' and that T_W vanishes in $V \setminus V_\epsilon$, we find with the help of Stoke's theorem

$$V \cdot W = \lim_{\epsilon \to 0} \int_{V \setminus V_\epsilon} T' = -\lim_{\epsilon \to 0} \int_{V \setminus V_\epsilon} d(\rho S) = \lim_{\epsilon \to 0} \int_{\partial V_\epsilon} \rho S = \lim_{\epsilon \to 0} \int_{\partial V_\epsilon} S.$$

The projection $V_\epsilon \to \{||z|| < \epsilon,\ w = 0\}$ is a $m_{p_0}(V, W)$-sheeted covering and, hence,

$$V \cdot W = \lim_{\epsilon \to 0} \int_{\partial V_\epsilon} S = m_{p_0}(V, W) \cdot \lim_{\epsilon \to 0} \int_{\partial B_\epsilon} \tilde{k}_{BM} = m_{p_0}(V, W) \cdot \lim_{\epsilon \to 0} \int_{B_\epsilon} \delta = m_{p_0}(V, W).$$

Here, after the projection we end up with the usual Bochner-Martinelli form (in p coordinates) whose differential is the dirac delta distribution, i.e. we forget that S is defined for w coordinates. This proves the theorem for the case in which W is smooth. For the general case, we use the following trick. Let Δ denote the diagonal in $M \times M$. It is a standard result that $V \cdot W = (V \times W) \cdot \Delta$. As the diagonal is smooth, we entered the first case and obtain

$$V \cdot W = \sum_{p \in V \cap W} m_{(p,p)}(V \times W, \Delta) = \sum_{p \in V \cap W} m_p(V, W).$$

$\square$

3 The Proper Mapping Theorem

In this chapter, we mean to prove the Proper Mapping Theorem, which can be stated as follows.

Theorem 3.1 (Proper Mapping Theorem). *Let M and N be complex manifolds and $V \subset M$ an analytic subvariety. Suppose we are given a holomorphic map $f\colon M \to N$ such that its restriction to V is proper. Then $f(V)$ is an analytic subvariety of N of dimension at most* $\dim(V)$.

We will prove this theorem under an additional technical hypothesis that is satisfied in most cases, namely:

Theorem 3.2 (Proper Mapping Theorem, version 2). *Let M, N, V, and f be as above. Assume that for every regular point $p \in V$ and every k-dimensional plane $\Lambda_p \subset T_pV$, $k \leq \dim(V)$, there exists a k-dimensional analytic subvariety Z of V with tangent space Λ_p at p. Then $f(V)$ is an analytic subvariety of N of dimension at most* $\dim(V)$.

Before we can prove the Proper Mapping Theorem, we need to recapture some notions from algebraic geometry. We begin by introducing Divisors and Line Bundles and study their interplay.

3.1 Divisors and Line Bundles

Divisors

Let M be a complex manifold and denote by $\mathcal{V}$ the set of irreducible analytic hypersurfaces in M. A *divisor* D on M is a formal sum (over $\mathbb{Z}$)

$$D = \sum_{V \in \mathcal{V}} \alpha_V V$$

that is locally finite in the sense that for any point $p \in M$ there exists a small neighborhood U such that for all but finitely many $V \in \mathcal{V}$ intersecting U we have $\alpha_V = 0$. We denote by $\mathrm{Div}(M)$ the set of all divisors on M. Given an analytic hypersurface W we can associate to it the divisor defined by $\alpha_V = 1$ if V is an irreducible component of W and $\alpha_V = 0$ otherwise. We can also associate divisors to functions on M. Suppose f is a holomorphic function defined in a neighborhood of a point $p \in M$. Given $V \in \mathcal{V}$ containing p, take a locally defining function g for V near p. Since the ring $\mathcal{O}_p$ of holomorphic functions near a given point p is a unique factorization domain and since g is irreducible, there exists some integer $a \geq 0$ and a holomorphic function h with $f = g^a h$ near p. We can take the largest possible such a (i.e. by requiring g and h to be relatively prime in $\mathcal{O}_p$) and call it the *order of f along V at p*.

Lemma 3.3. *With V and f as above, the order of f along V at p is well-defined in that it is independent of the choice of g. Moreover, it is independent of the point $p \in V$.*

Proof. The first statement is immediate: Suppose $f = g_1^{a_1} h_1 = g_2^{a_2} h_2$ with a_1 and a_2 maximal. We can also write $g_2 = g_1^b h_3$ so that $f = g_1^{a_2+b} h_3^{a_2} h_2$. a_1 being maximal implies $a_1 \geq a_2 + b \geq a_2$ and, by symmetry, $a_2 \geq a_1$. Thus, the order is well-defined. For the second statement, it suffices to show that the order of f along V at p remains the same in an open neighborhood of p since V is irreducible. The definition of the order required g and h to be relatively prime in $\mathcal{O}_p$. In order to conclude, we will show that such elements remain relatively prime in $\mathcal{O}_q$, where q is any point close to p. As before, by taking suitable coordinates, we may assume that each function is the product of a Weierstrass polynomial in the last coordinate z_n and a non-vanishing holomorphic function. Write $g = P_1 k_1$ and $h = P_2 k_2$. Suppose j is a divisor of both g and h in $\mathcal{O}_q$. g and h being relatively prime implies that P_1 and P_2 are relatively prime in the ring of Weierstrass polynomials. Since the latter is a unique factorization domain, we can take a linear combination of P_1 and P_2 to obtain a holomorphic function that depends only on the first

$n-1$ coordinates. As k_1 and k_2 are non-vanishing, we also get a linear combination of g and h that depends only on the first $n-1$ coordinates. But then j is also a divisor of this function and can therefore itself depend only on the first $n-1$ coordinates. If g and h do not have a common zero in the domain of j, then g and h are obviously relatively prime. Otherwise, j also has a zero w. Thus, $j(w_0, \ldots, w_{n-1}, z_n)$ is constantly zero in z_n implying that also $P_1(w_0, \ldots, w_{n-1}, z_n)$ and $P_2(w_0, \ldots, w_{n-1}, z_n)$ are constantly zero in z_n. Hence, P_1 and P_2 are the zero polynomial. In particular, f is constantly zero. □

By the lemma, we can therefore speak of the *order of f along V*, which we denote by $\mathrm{ord}_V(f)$. If we do not begin with a holomorphic function but instead with a meromorphic function $f = f_1/f_2$, then we define $\mathrm{ord}_V(f) = \mathrm{ord}_V(f_1) - \mathrm{ord}_V(f_2)$. Using the order, we can associate to every meromorphic function f on M a divisor by

$$(f) = \sum_{V \in \mathcal{V}} \mathrm{ord}_V(f) V.$$

For $f = f_1/f_2$ with f_1 and f_2 relatively prime, we call $(f)_0 = (f_1)$ the *divisor of zeros* and $(f)_\infty = (f_2)$ the *divisor of poles*. Note that locally any divisor can be seen as the divisor of a meromorphic function. Indeed, in a neighborhood U of a given point only finitely many $\alpha_{V_1}, \ldots, \alpha_{V_m}$ are non-zero and if g_j denotes a locally defining function of V_j, then

$$\left(g_1^{\alpha_{V_1}} \cdots g_m^{\alpha_{V_m}}\right) = \sum_{j=1}^{m} \alpha_{V_j} V_j = \sum_{V \in \mathcal{V},\ V \cap U \neq \emptyset} \alpha_V V.$$

However, this product might not give rise to a well-defined global function. Facing such a local-global problem, sheaf theory comes to mind. The divisor associated to a function remains unchanged after multiplication with a non-zero holomorphic function. Moreover, by multiplying with non-zero holomorphic functions, the above products agree on the intersection of their domains. Thus, a divisor is, in fact, a global section of the quotient sheaf $\mathcal{M}^*/\mathcal{O}^*$, where $\mathcal{M}^*$ is the multiplicative sheaf of meromorphic functions not identically zero and $\mathcal{O}^*$ is the multiplicative sheaf of non-zero holomorphic functions. Conversely, given an element in $\mathcal{M}^*/\mathcal{O}^*(M)$, we get an associated divisor. In particular, by proposition 1.6, we have isomorphisms

$$\mathrm{Div}(M) \cong \mathcal{M}^*/\mathcal{O}^*(M) \cong \check{\mathrm{H}}^0(M, \mathcal{M}^*/\mathcal{O}^*).$$

Line Bundles

A *line bundle* on M is a smooth, complex, rank 1, vector bundle on M. In this section, we are interested in holomorphic line bundles, that is line bundles $\pi \colon L \to M$ where L admits a complex manifold structure and the trivialization maps are biholomorphic. Recall that this involves an open cover $\underline{U} = \{U_\alpha\}_\alpha$ of M, local trivializations

$$\phi_\alpha \colon \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{C},$$

and corresponding transition functions

$$g_{\alpha,\beta} \colon U_\alpha \cap U_\beta \to \mathrm{GL}(1, \mathbb{C}) \cong \mathbb{C}^*,\ z \mapsto \left(\phi_\alpha \circ \phi_\beta^{-1}\right)\Big|_{\{z\} \times \mathbb{C}}.$$

Note that $1/g_{\alpha,\beta} = g_{\beta,\alpha}$. Suppose we are given another trivialization $\tilde{\phi}_\alpha$ over the same open cover. Since on each fibre ϕ_α and $\tilde{\phi}_\alpha$ are linear isomorphisms to $\mathbb{C}$, there exists a function $f_\alpha \colon U_\alpha \to \mathbb{C}^*$ with

$$\tilde{\phi}_\alpha|_{\pi^{-1}(z)} = f_\alpha(z) \cdot \phi_\alpha|_{\pi^{-1}(z)}$$

for every $z \in U_\alpha$. This function is holomorphic since ϕ_α and $\tilde{\phi}_\alpha$ are. Thus, for the transition functions we get $\tilde{g}_{\alpha,\beta} = (f_\alpha/f_\beta) \cdot g_{\alpha,\beta}$. Conversely, any collection of maps of the form $f_\alpha \cdot \phi_\alpha$, with $f_\alpha \colon U_\alpha \to \mathbb{C}^*$ holomorphic, defines a trivialization of the same line bundle $L \to M$. Hence, transition functions $(g_{\alpha,\beta})_{(\alpha,\beta)}$ and $(\tilde{g}_{\alpha,\beta})_{(\alpha,\beta)}$ give rise to the same line bundle if and only if $\tilde{g}_{\alpha,\beta} = (f_\alpha/f_\beta) \cdot g_{\alpha,\beta}$ for some collection of holomorphic maps $f_\alpha \colon U_\alpha \to \mathbb{C}^*$. We can also cast holomorpic line bundles into the light of sheaf theory. Given a line bundle as above, note that $\sigma = (g_{\alpha,\beta})_{(\alpha,\beta)}$ defines an element in the cochain complex $\mathrm{C}^1(\underline{U}, \mathcal{O}^*)$. Moreover,

$$(\delta\sigma)_{(\alpha,\beta,\gamma)} = g_{\beta,\gamma} \cdot \frac{1}{g_{\alpha,\gamma}} \cdot g_{\alpha,\beta} = 1,$$

so σ is a cocycle. Conversely, any cocycle $\sigma = (g_{\alpha,\beta})_{(\alpha,\beta)} \in \mathrm{C}^1(\underline{U}, \mathcal{O}^*)$ characterizes a unique holomorphic line bundle. Furthermore, σ and $\tilde{\sigma}$ define the same cohomology class in $\mathrm{H}^1(\underline{U}, \mathcal{O}^*)$ if and only if $\sigma \cdot \tilde{\sigma}^{-1} = (g_{\alpha,\beta}/\tilde{g}_{\alpha,\beta})_{(\alpha,\beta)}$ equals a coboundary $\delta\tau$, for some $\tau = (f_\alpha)_\alpha \in \mathrm{C}^0(\underline{U}, \mathcal{O}^*)$. This is the case if and only if $g_{\alpha,\beta}/\tilde{g}_{\alpha,\beta} = f_\beta/f_\alpha$ on every $U_\alpha \cap U_\beta$ and we have just seen that this holds if and only if σ and $\tilde{\sigma}$ define the same line bundle. We conclude that there is an identification of the set of holomorphic line bundles on M modeled on the open cover $\underline{U}$ with the cohomology group $\mathrm{H}^1(\underline{U}, \mathcal{O}^*)$. By taking finer and finer coverings, we obtain an identification of the set of holmorphic line bundles, which we denote by $\mathrm{LB}_h(M)$, on M with the Čheck cohomology group $\check{\mathrm{H}}^1(M, \mathcal{O}^*)$. Under this identification, $\mathrm{LB}_h(M)$ obtains a group structure, which corresponds to taking the tensor product of two bundles as the group operation and taking the dual bundle as the inverse element.

Next, we want to show that any divisor comes with an associated line bundle. As discussed above, a given divisor D always has a local representation by a meromorphic function. Take an open cover $\underline{U} = \{U_\alpha\}_\alpha$ such that $D = (f_\alpha)$ in U_α with f_α a meromorphic function on U_α. Since f_α is given by powers of locally defining maps of elements $V \in \mathcal{V}$, any quotient $g_{\alpha,\beta} = f_\alpha/f_\beta$ is an element of $\mathcal{O}^*(U_\alpha \cap U_\beta)$. Clearly, $(g_{\alpha,\beta})_{(\alpha,\beta)}$ is a cocycle and, hence, gives rise to some line bundle. If we had taken different representatives $\tilde{f}_\alpha$ of the divisor D, then $h_\alpha = \tilde{f}_\alpha/f_\alpha$ is in $\mathcal{O}^*(U_\alpha)$ and, hence, $\tilde{g}_{\alpha,\beta} = (h_\alpha/h_\beta) \cdot g_{\alpha,\beta}$ shows that the different representatives give rise to the same line bundle. Thus, we may write $[D]$ for the line bundle associated to a divisor. Note that, by additivity of the order of a map along a hypersurface, if f_α and $\tilde{f}_\alpha$ are local representations of divisors D and $\tilde{D}$, respectively, then $D + \tilde{D}$ has local representations $f_\alpha \cdot \tilde{f}_\alpha$. Therefore, the map that sends a divisor to its associated line bundle is, in fact, a group homomorphism from $\mathrm{Div}(M)$ to $\mathrm{LB}_h(M)$. In the special case where $D = (f)$ is associated to a meromorphic function on M, its local representation over any open cover is given by the restrictions of the map f. In particular, the associated line bundle of (f) is trivial. Conversely, if $[D]$ is trivial and D is represented by f_α, then there are functions $h_\alpha \in \mathcal{O}^*(U_\alpha)$ with $f_\alpha/f_\beta = g_{\alpha,\beta} = h_\alpha/h_\beta$. Then the local sections $f_\alpha/h_\alpha = f_\beta/h_\beta$ glue to a global section $f \in \mathcal{M}^*(M)$ and $D = (f)$. This establishes the following proposition.

Proposition 3.4. *The line bundle associated to a divisor is trivial if and only if the divisor is the divisor associated to a meromorphic function.*

3.2 The Levi Extension Theorem

The most important result we need to prove the Proper Mapping Theorem is the Levi Extension Theorem, which allows us to extend meromorphic functions over subvarieties. We begin with the local version about extending subvarieties.

Theorem 3.5 (Levi Extension Theorem, local version)**.** *Let V be an analytic subvariety in the polycylinder $\Delta^n \subset \mathbb{C}^n$, $n \geq 2$, of (complex) codimension at least 2, and let $D \subset \Delta^n \setminus V$ be a subvariety of (complex) codimension 1. Then the closure of D in Δ^n is analytic.*

Here is one preliminary result we need for the proof. Let Δ denote a polycylinder in $\mathbb{C}$ and Δ^* that same polycylinder punctured in the origin. Define $\Delta' = \Delta^* \times \Delta$.

Lemma 3.6. *The cohomlogy group* $\check{\mathrm{H}}^1(\Delta', \mathcal{O}^*)$ *vanishes.*

Proof. The exponential sequence of sheaves

$$0 \to \mathbb{Z}_{\Delta'} \hookrightarrow \mathcal{O} \xrightarrow{\exp} \mathcal{O}^* \to 0$$

induces a long exact sequence in cohomology

$$\cdots \to \check{\mathrm{H}}^1(\Delta', \mathcal{O}) \to \check{\mathrm{H}}^1(\Delta', \mathcal{O}^*) \to \check{\mathrm{H}}^2(\Delta', \mathbb{Z}_{\Delta'}) \to \cdots.$$

The term on the right is just singular cohomology with coefficients in $\mathbb{Z}$, which vanishes by the Künneth formula. Thus, it suffices to show that $\check{\mathrm{H}}^1(\Delta', \mathcal{O})$ vanishes. By the Dolbeault theorem, this cohomology group is isomorphic to $\mathrm{H}^{0,1}_{\bar\partial}(\Delta')$. Let $B(\epsilon, r)$ denote the annulus around the origin in $\mathbb{C}$ and take a smooth bump function ρ that is 1 on $B(\epsilon, r)$ and 0 close to the origin. Abbreviate $\Delta'(\epsilon, r) = B(\epsilon, r) \times \Delta$. Suppose we are given a $\bar\partial$-closed $(0,1)$-form $\psi = h_1 d\bar z_1 + h_2 d\bar z_2$ on Δ'. Define a function on $\Delta'(0, r)$ by

$$f(z_1, z_2) = \frac{1}{2\pi i} \int_{|w| \le r} \frac{\rho(w) h_1(w, z_2)}{w - z_1} dw \wedge d\bar w.$$

By the Cauchy-type lemma 2.17, we then have $\frac{\partial}{\partial \bar z_1} f(z) = \rho(z_1) h_1(z)$ and, hence,

$$\rho\psi - \bar\partial f = \left(\rho(z_1) h_2(z) - \frac{\partial}{\partial \bar z_2} f(z) \right) d\bar z_2.$$

Denote the function in the brackets by ϕ. Since ψ is $\bar\partial$-closed, the function ϕ is holomorphic in z_1 in $\Delta'(\epsilon, r)$. Thus, it has a power series expansion in $\Delta'(\epsilon, r)$ in the first variable,

$$\phi(z) = \sum_{k \ge 0} a_k(z_2) z_1^k.$$

Now we isolated the z_1-dependence and obtained forms $a_k(z_2) d\bar z_2$ in $\mathrm{H}^{0,1}_{\bar\partial}(\Delta) = 0$. Thus, we find functions f_k on Δ whose anti-holomorphic derivatives are the functions a_k. Consequently,

$$\psi - \bar\partial f = \sum_{k \ge 0} z_1^k \frac{\partial}{\partial \bar z_2} f_k(z_2) d\bar z_2 = \bar\partial \left(\sum_{k \ge 0} z_1^k f_k(z_2) \right)$$

on $\Delta'(\epsilon, r)$. Now take a sequence $(\epsilon_m, r_m) \to (0, 1)$. We have proved that we can find a function g_m with $\bar\partial g_m = \psi$ on $\Delta'(\epsilon_m, r_m)$. We can also pick a function α with $\bar\partial \alpha = \psi$ on $\Delta'(\epsilon_{m+1}, r_{m+1})$. Then $g_m - \alpha$ is holomorphic in $\Delta'(\epsilon_m, r_m)$ and, hence, has a power series expansion. Truncate the expansion to obtain a polynomial β with

$$\sup_{\Delta'(\epsilon_{m-1}, r_{m-1})} |(g_m - \alpha) - \beta| < \frac{1}{2^m}.$$

Then define $g_{m+1} = \alpha + \beta$. This way, $\bar\partial g_{m+1} = \psi$ in $\Delta'(\epsilon_{m+1}, r_{m+1})$ and $|g_{m+1} - g_m|$ is uniformly bounded in $\Delta'(\epsilon_{m-1}, r_{m-1})$ by 2^{-m}. In particular, the limit g of g_m exists and is a solution to $\bar\partial g = \psi$ in Δ'. This finishes the lemma. ☐

Note that the arguments in the proof of the lemma still work for polycylinder of larger dimensions. In particular, $\check{\mathrm{H}}^1(\mathbb{C}^* \times \mathbb{C}^n, \mathcal{O})$ is always zero for any $n \ge 1$.

Proof of theorem 3.5. We first tend to the simplified case where $n = 2$ and V is the origin. We wish to show that $D^* = D \cap \Delta'$ has a globally defining function. Recall that we identified $\check{\mathrm{H}}^1(\Delta', \mathcal{O}^*)$ with the set of holomorphic line bundles on Δ'. The lemma tells us that there are only trivial line bundles. By proposition 3.4, $D^* = D \cap \Delta'$ is the divisor of some meromorphic function h on Δ'. This function actually cannot have poles since D is a subvariety. Thus, h is holomorphic. Since D has codimension 1, we may assume (up to rotating the axes) that it does not contain $\{z_1 = 0\}$. Then D^* and $\{z_1 = 0\}$ intersect only in finitely many points and we can find δ and ϵ such that the set $\{|z_1| \leq \delta,\ |z_2| = \epsilon\}$ does not intersect D. Thus, for any $z = (z_1, z_2)$ with $0 < |z_1| \leq \delta$, the "winding number"

$$\frac{1}{2\pi i} \int_{|z_2|=\epsilon} \frac{dh(z_1, z_2)}{h(z_1, z_2)}$$

is well-defined. Since it is integer-valued and depends continuously on z_1, it is constant in z_1. Let $d \in \mathbb{Z}$ denote its value. Note that the zeros are simple and, therefore, there are d-many zeroes $\{z_1, z_{2,\nu}(z_1)\}_{1 \leq \nu \leq d}$, for each z_1. The sum

$$\phi_j(z_1) = \sum_{\nu=1}^{d} z_{2,\nu}(z_1)^j,$$

where $j \geq 0$, is holomorphic in z_1 since we can use the Residue theorem to write it as the integral

$$\frac{1}{2\pi i} \int_{|z_2|=\epsilon} z_2^j \frac{dh(z_1, z_2)}{h(z_1, z_2)}.$$

Boundedness of the functions ϕ_j implies that the origin is a removable singularity. Let

$$\sigma_m(z_1) = \sum_{1 \leq k_1 < \cdots < k_m \leq d} \prod_{i=1}^{m} z_{2,k_i}(z_1),$$

$1 \leq m \leq d$, be the elementary symmetric polynomials for $z_{2,1}(z_1), \ldots, z_{2,d}(z_1)$. Set

$$F(z_1, z_2) = z_2^d - \sigma_1(z_1) z_2^{d-1} + \cdots + (-1)^d \sigma_d(z_1) = \prod_{\nu=1}^{d} (z_2 - z_{2,\nu}(z_1)).$$

Since we can rewrite the $\sigma_1, \ldots, \sigma_d$ as polynomial expressions in $\phi_1, \ldots, \phi_d$, the function F is holomorphic and extends to $\{z_1 = 0\}$. As the roots of F for $z_1 \neq 0$ are exactly $(z_1, z_{2,\nu}(z_1))$, $1 \leq \nu \leq d$, the divisor of F is exactly $\overline{D}$. This finishes the special case $n = 2$ and $V = \{0\}$. If n is arbitrary and V a linear subspace of $\mathbb{C}^n$, then the argument is analogous. Since the problem is local, this proves the theorem for regular varieties V. Fortunately, this is sufficient: by the above, we can extend D to $\Delta \setminus V_s$, where V_s denotes the subvariety of V of singular points. By applying the argument to V_s, we can also extend D to $\Delta \setminus (V_s)_s$. Continuing this procedure yields the theorem. □

The global version about meromorphic functions follows easily from the local version once we observed Hartog's theorem, which displays one of the main strengths of multi-variable complex analysis as opposed to complex analysis of one variable.

Theorem 3.7 (Hartog)**.** *Let $U \subset \mathbb{C}^n$ be a polycylinder, $n \geq 2$, and $V \subset \mathbb{C}^n$ a subvariety of (complex) codimension at least 2. Any holomorphic function on $U \setminus V$ extends to a holomorphic function on U.*

Proof. As the codimension of V in U is at least 2, there are some coordinates for which $\{|z_n| = r\} \subset \partial U$ does not intersect V. Given a holomorphic function f on $U \setminus V$, simply define

$$F(z_1, \ldots, z_n) = \frac{1}{2\pi i} \int_{|w_n|=r} \frac{f(z_1, \ldots, z_{n-1}, w_n)}{w_n - z_n} dw_n,$$

which equals f on $U \setminus V$ by Cauchy's formula. □

Theorem 3.8 (Levi Extension Theorem, global version)**.** *Let M be a complex manifold of (complex) dimension at least 2 and $V \subset M$ an analytic subvariety of (complex) codimension at least 2. Suppose f is a meromorphic function defined on $M \setminus V$. Then f extends to a meromorphic function on M.*

Proof. Consider the polar divisor $(f)_\infty$. By the local version of the Levi Extension Theorem, its closure is an analytic subvariety of M. Given a point p with a small neighborhood U, take a locally defining function g of this subvariety. Then the product $f \cdot g$ is holomorphic in $U \setminus V$ by construction. By Hartog's theorem, $f \cdot g$ extends to a holomorphic function h in U. A meromorphic extension of f is therefore given by h/g. □

3.3 The Proof of the Proper Mapping Theorem

We will prove the theorem by induction on $m = \dim(V)$. If $m = 0$, V is just a collection of isolated points. Then $f(V)$ also consists of isolated points as $f|_V$ is proper. Now assume that the theorem is proved for all analytic subvarieties of dimension less than m. We will first prove a special case and argue afterwards how to reduce the general theorem to this instant.

Step 1. *Suppose $N = \Delta$ is the polycylinder in $\mathbb{C}^{m+1}$. Define a $(1,1)$-current by integration of pullback,*

$$S\colon \Omega_c^{m,m}(\Delta) \to \mathbb{C},\ S(\phi) = \int_{V^*} f^*\phi.$$

This is well-defined as f is proper. Moreover, the appearance of the pullback does not invalidate the arguments in the proof of lemma 2.16 (that integration over a subvariety defines a closed positive current), so the current S is also closed and positive.

A posteriori, once the Proper Mapping Theorem is proved, this current must be the one associated to the subvariety $f(V)$ (up to a constant factor, namely the degree of f). Here is the heart of of the proof:

Step 2. *Suppose V is irreducible and $N = \Delta$ is the polycylinder in $\mathbb{C}^{m+1}$. Furthermore, assume that f has maximal rank m at some point in V^*. Then $f(V)$ is an analytic subvariety in Δ of dimension m.*

Proof. Let $W \subset V$ denote the union of the set of singular points and the set of those points where f does not have maximal rank. Then W is an analytic subvariety of V of at least one (complex) dimension less. Hence, by the induction hypothesis, $f(W)$ is an analytic subvariety and its dimension is at most $m-1$. At any point $p \in V \setminus W$, f has maximal rank and, hence, is locally invertible with holomorphic inverse. The image of a small neighborhood of p will therefore be an open subset of an analytic subvariety of dimension m. Thus, both $f(V \setminus W)$ and $f(W)$ are analytic subvarieties and we must show that their union defines a single subvariety. Let us inspect a neighborhood U of a point lying in $f(W)$. We can apply the $\partial\bar{\partial}$-Poincaré lemma to the current from step 1 to find that $S = i\partial\bar{\partial}\rho$ in U for some real function ρ. Given a point $q \in f(V \setminus W) \cap U$, take a locally defining holomorphic function h of the latter. By the Poincaré Lelong equation, the current T associated to the subvariety $f(V \setminus W)$ is locally equal to $\frac{i}{\pi}\partial\bar{\partial}\log|h|$. If k_0 denotes the degree of the map f, then the degree theorem for integration yields near q

$$0 = S(\phi) - k_0 T(\phi) = i\partial\bar{\partial}\left(\rho - \frac{k_0}{\pi}\log|h|\right).$$

Thus, $\rho - \frac{k_0}{\pi}\log|h|$ is the real part $\Re(g)$ of a holomorphic function g in a neighborhood of q. Note that $d\log h = \partial \log|h|^2$ since h is holomorphic. Thus, we get the equality of derivatives

$$dg = 2\partial\Re(g) = 2\partial\rho - \frac{k_0}{\pi} d\log h.$$

In particular, $\partial\rho$ is closed in $U \setminus f(V)$. The set $f(V \setminus W)$ is exactly the set of poles of $\partial\rho$ as the appearance of $\log h$ shows. By the regularity of the $\bar{\partial}$-operator, $\partial\rho$ is holomorphic in $U \setminus f(V)$. Thus, its coefficient functions are meromorphic functions on $U \setminus f(W)$. As $f(W)$ has (complex) dimension at least two smaller than Δ, the Levi Extension Theorem is applicable and $\partial\rho$ extends to U. Then $\overline{f(V \setminus W)} = f(V \setminus W) \cup f(W)$ is the polar divisor of $\partial\rho$, so the proof is finished. □

It remains to perform the reductions. We will begin by justifying the additional hypothesis we put on the analytic subvariety V and on the function f.

Step 3. *It suffices to prove the Proper Mapping Theorem 3.2 for irreducible V and for functions f for which there exists a smooth point at which f has maximal rank $m = \dim(V)$.*

Proof. Since $f|_V$ is proper, given any compact subset of N only finitely many components $V_1, \ldots, V_r$ of V will intersect its preimage. If we prove that $f(V_1), \ldots, f(V_r)$ are analytic subvarieties, then (locally) $f(V)$ is a finite union of analytic subvarieties, hence itself an analytic subvariety. Therefore, we may assume that V is irreducible. Now pick a smooth point p_0 where f has maximal rank $k \leq m$. If $k = m$, we are done. Otherwise, the additional technical hypothesis in theorem 3.2 provides us with an analytic subvariety $Z \subset V$ with tangent space at p_0 the plane on which f has full rank. The lemma is proved if we show $f(Z) = f(V)$. We may assume that $f|_Z$ has full rank in a neighborhood of p_0. By the implicit function theorem, there is a $m - k$ dimensional submanifold of V (the graph of the implicit function) going through p_0 on which f has value $f(p_0)$. Similarly, there are $(m-k)$-submanifolds associated to points p in Z near p_0 that get mapped to $f(p)$. These submanifolds foliate an open subset W of V, which satisfies $f(W) = f(W \cap Z)$. We conclude this step with the identity principle and irreducibility of V. □

Next, we turn to the additional hypothesis on N. As the question of whether $f(V)$ is an analytic subvariety of N is a local question, this reduces us to the case in which N is a polycylinder in $\mathbb{C}^n$. We need to ensure that its dimension can be taken to be $n = m + 1$. The key step to this is:

Step 4. *Suppose N is a polycylinder in $\mathbb{C}^n$ and $\pi \colon N \to \Delta^{m+1}$ is a projection onto a polycylinder of smaller dimension. Then the restriction of π to $f(V)$ is proper.*

Proof. For every $w \in \Delta$, let Λ_w denote the hyperplane $\{z_1 = w\}$ in $\mathbb{C}^n$. We get a family of new subvarieties of V of lesser dimension by $V_w = V \cap f^{-1}(\Lambda_w)$. By the induction hypothesis, each $f(V_w)$ is an analytic subvariety of dimension at most $m - 1$. As subvarieties are closed subsets, the projection of each $f(V_w)$ onto Δ^m is proper. Now given a compact subset K of Δ^{m+1}, denote by K_w the intersection $(\{w\} \times \Delta^m) \cap K$. Let $K_1 \subset \Delta$ denote the compact subset obtained from K by projecting onto the first coordinate. Then

$$\left(\pi|_{f(V)}\right)^{-1}(K) = \bigcup_{w \in K_1} \left(\pi|_{f(V_w)}\right)^{-1}(K_w)$$

and each set in the union is compact. In fact, the union itself is compact since the sets in the union are contained in parallel hyperplanes. Hence, $\pi|_{f(V)}$ is proper. □

We have gathered all the necessary ingredients. All that is left to do is fit the pieces together.

Step 5. *The Proper Mapping Theorem holds in full generality.*

Proof. As discussed, we may assume that N is a polycylinder in $\mathbb{C}^n$. By the last step, the composition $\pi \circ f \colon N \to \Delta^{m+1}$ is proper for any choice of smaller polycylinder $\Delta^{m+1} \subset N$. By steps 2 and 3, such $\pi \circ f(V)$ is an analytic subvariety of dimension at most m. Note that we only need finitely many different choices of such polycylinder to reconstruct $f(V)$ from the various $\pi \circ f(V)$. In particular, collecting locally defining functions of each $\pi \circ f(V)$ and composing them with the corresponding projection gives a set of locally defining functions for $f(V)$. Thus, $f(V)$ is itself an analytic subvariety. If it had dimension greater than m, then we could pick an $m+1$-dimensional polycylinder such that the projection onto it could not have dimension at most m. This finishes the proof of the Proper Mapping Theorem. $\square$

References

[1] Bredon, G. E. *Sheaf theory*, second ed., vol. 170 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1997.

[2] Cannas da Silva, A. *Lectures on symplectic geometry*, vol. 1764 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2008.

[3] Griffiths, P., and Harris, J. *Principles of algebraic geometry*. Wiley Classics Library. John Wiley & Sons, Inc., New York, 1994. Reprint of the 1978 original.

[4] Hartshorne, R. *Algebraic geometry*. Springer-Verlag, New York-Heidelberg, 1977. Graduate Texts in Mathematics, No. 52.

[5] Laurent-Thiébaut, C. *Holomorphic function theory in several variables*. Universitext. Springer-Verlag London, Ltd., London; EDP Sciences, Les Ulis, 2011. An introduction, Translated from the 1997 French original.

[6] Nicolaescu, L. I. *Lectures on the geometry of manifolds*, second ed. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2007.