Christmas Stars

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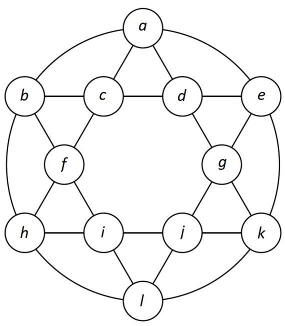
Abstract

We count the number of distinct hexagonal Christmas stars decorated by a combinatorialist with the first twelve natural numbers. To do so, we first eliminate symmetry from the problem by considering an appropriate group action on the star, and then we count the number of induced equivalence classes.

1. Combinatorial Christmas stars

The problem asks to fill the star shown below with the numbers 1 through 12 such that along all edges and along the outer circle the numbers sum to 26. There are several possible solutions. Naturally, the question arises exactly how many different solutions there are.



$x = (a, \ldots, l)$ is a solution of the star if $\{a, \ldots, l\} = \{1, \ldots, 12\}$ and

- $a + b + e + h + k + l = 26$,
- $a + c + f + h = 26$,
- $a + d + g + k = 26$,
- $h + i + j + k = 26$,
- $b + c + d + e = 26$,
- $b + f + i + l = 26$,
- $e + g + j + l = 26$.

Theorem 1. *There are exactly 72 different solutions.*

To make the theorem more accessible, we consider an appropriate group action on the star. Let S denote a rotation of the star by angle $\pi/3$ and let R denote a reflection along the vertical axis. The group generated by R and S is the dihedral group D_6. It is easy to see that applying R or S to a solution of the star yields another solution. Hence, there is a well-defined action of D_6 on the set of all possible solutions. Any group action defines an equivalence relation by calling two solutions equivalent if they lie on the same orbit. Since this group action clearly is free and D_6 has cardinality 12, every equivalence class contains exactly 12 representatives. Thus, the total number of solutions to the star is the number of equivalence classes times 12. In conclusion, the theorem reduces to the following counting problem.

Proposition 2. *There are exactly six equivalence classes.*

2. Counting equivalence classes

Suppose $x = (a, \ldots, l)$ is a solution. Summing up the three edges of each triangle, we find

$$2a + 2h + 2k + c + d + f + g + i + j = 78 = 2b + 2e + 2l + c + d + f + g + i + j.$$

Since also $26 = a + b + e + h + k + l$, we obtain $a + h + k = 13 = b + e + l$, that is the corners of each triangle always sum to 13. The only triples of numbers in $\{1, \ldots, 12\}$ that sum to 13 are

$$(1, 2, 10), (1, 3, 9), (1, 4, 8), (2, 3, 8), (1, 5, 7), (2, 4, 7), (2, 5, 6), (3, 4, 6).$$

Since we need two such triples (one for the set of corners of each of the two triangles) with distinct numbers, the only options for the corners of the triangles are

$$(1,2,10)+(3,4,6),\qquad (1,3,9)+(2,4,7),\qquad (1,3,9)+(2,5,6),$$
$$(1,4,8)+(2,5,6),\qquad (2,3,8)+(1,5,7),\qquad (1,5,7)+(3,4,6).$$

Note that any element $g \in D_6$ leaves the outer circle of the star invariant. Thus, if two solutions have different pairs of triples describing their triangle corners, then they cannot be equivalent.

Now, we argue that for two of those pairs of triples, no solution with the corresponding triangle corners can exist. The first pair is $(1,3,9)+(2,5,6)$. Suppose for contradiction that x is a solution such that the corners of the triangles match these two triples. Since x is a solution, the two numbers between 1 and 3 must be 10 and 12; see the left star in Fig. 1. The rest of the inner circle must be filled with $\{4,7,8,11\}$. However, the two numbers between 1 and 9 must sum to 16, which is impossible to achieve.

Secondly, suppose x is a solution such that the corners of the triangles match $(1,5,7)+(3,4,6)$. Note that there is an element $g \in D_6$ such that the position of the 1 in gx is at the top and the position of the 5 in gx is at the lower left. Since gx is a solution, the two numbers between the 5 and the 7 (i.e. i and j) must be 2 and 12, and between the 1 and the 5 (i.e. c and f) must be 9 and 11; see the right star in Fig. 1. This is impossible for the following reason: if the 2 and the 9 neighbor each other (meaning that they take positions f and i), then the corners at the b and l position must sum to 15, which cannot be since those corners are taken from $\{3,4,6\}$; if the 2 and the 1, the 12 and the 9, or the 12 and the 11 neighbor each other, then the b and l position also cannot yield the correct sum.

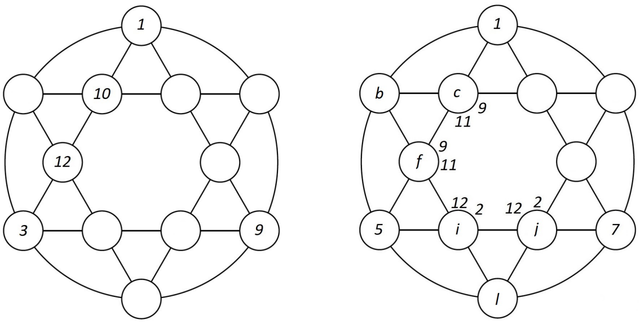


Figure 1: Impossibility for two pairs of triples specifying the corners of the triangles.

As we verify further below, for the other four pairs of triples, that is

$$(1,2,10)+(3,4,6),\qquad (1,3,9)+(2,4,7),\qquad (1,4,8)+(2,5,6),\qquad (2,3,8)+(1,5,7),$$

there is a solution for which the corners of the triangles are given by those two triples. There may even be more than one solution per pair of triple. The question arises when different solutions for the same pair give rise to different equivalence classes. Suppose x and x' are two such solutions. We can take elements $g, g' \in D_6$ such that all three corners of one of the triangles are the same for gx and $g'x'$. If $gx = g'x'$, then of course x and x' belong to the same equivalence class. If $gx \neq g'x'$, then there cannot be an element $g'' \in D_6$ such $gx = g''g'x'$, because g'' would keep at least three nodes of the star fixed (namely, the three corners of the above triangle) and this is only possible for the identity element in D_6. Hence, if $gx \neq g'x'$, then x and x' belong to different equivalence classes.

It remains to use this knowledge to find all possible solutions for each pair of triples. We begin with $(1,2,10)+(3,4,6)$. Suppose x is a solution with corners of the triangles given by this pair of triple. As above, we do not change the equivalence class by assuming that the 1 sits at the top and the 2 sits at the lower left. Since x is a solution, the two numbers between the 1 and the 2 must be 11 and 12. The 11 can sit either above or below the 12. We now show that each choice gives rise to exactly one equivalence class; see Fig. 2. If the 11 sits above, then its neighbor to the right must be the 7 or the 8 because the d and g position must sum to 15. In fact, it must be the 8 because the corners at positions b and e are taken from $\{3,4,6\}$ and cannot sum to 8. This fixes the 7 at position g. The 12 cannot be neighbored by the 9 because otherwise the corners b and l would have to sum to 3. This determines the 5, the 9, and finally also the 3, the 4, and the 6. This concludes the case where the 11 sits above the 12. Now, suppose the 11 sits below.

This does not change the fact that the 7 and the 8 have to be at positions d and g and the 5 and the 9 have to be at f and i. The 12 cannot neighbor the 8 because b and e cannot sum to 6. By the same reason, the 11 cannot neighbor the 9. This determines the remaining entries.

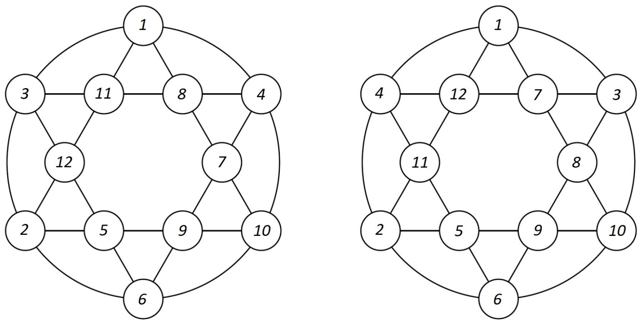


Figure 2: The only two solutions for the pair $(1, 2, 10) + (3, 4, 6)$.

For a solution corresponding to the pair $(1, 3, 9) + (2, 4, 7)$, we may again assume that the 1 sits at the top and the 3 sits at the lower left. The two numbers between the 1 and the 3 must be 10 and 12, between the 1 and the 9 must be 5 and 11, and between the 3 and the 9 must be 6 and 8; see the left star in Fig. 3. If the 10 sits below the 12, then the 10 cannot neighbor the 6 because the corners at b and l are taken from $\{2, 4, 7\}$ and cannot sum to 10, but neither can the 10 neighbor the 8 because b and l also cannot sum to 8. Thus, the 10 sits above the 12. By the same reason, the 12 cannot neighbor the 6 and the 10 cannot neighbor the 11. This fixes the remaining entries.

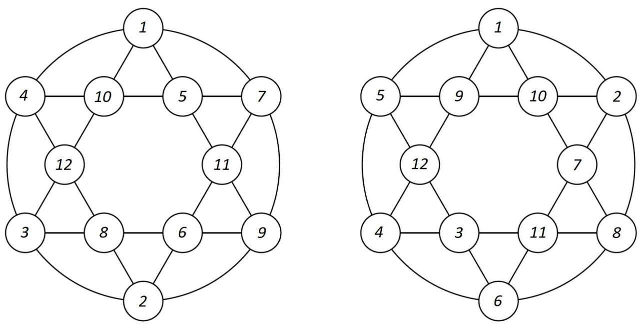


Figure 3: The only solution for the pair $(1, 3, 9) + (2, 4, 7)$ and $(1, 4, 8) + (2, 5, 6)$, respectively.

For the other two pairs, the arguments are completely analogous. The pair $(1, 4, 8) + (2, 5, 6)$ admits exactly one solution (see the right star in Fig. 3) and $(2, 3, 8) + (1, 5, 7)$ exactly two (see Fig. 4). This concludes counting the equivalence classes and finishes the proof of the theorem.

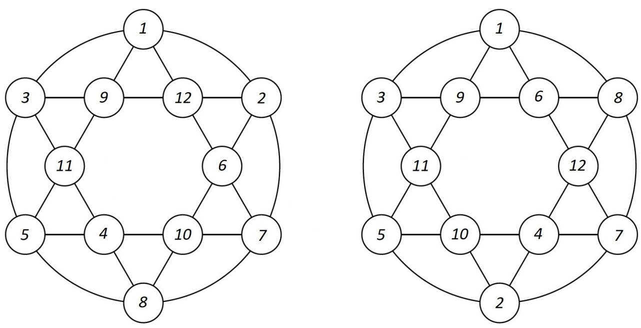


Figure 4: The only two solutions for the pair $(2, 3, 8) + (1, 5, 7)$.