Magnetic and Exotic Anosov Hamiltonian Structures

by

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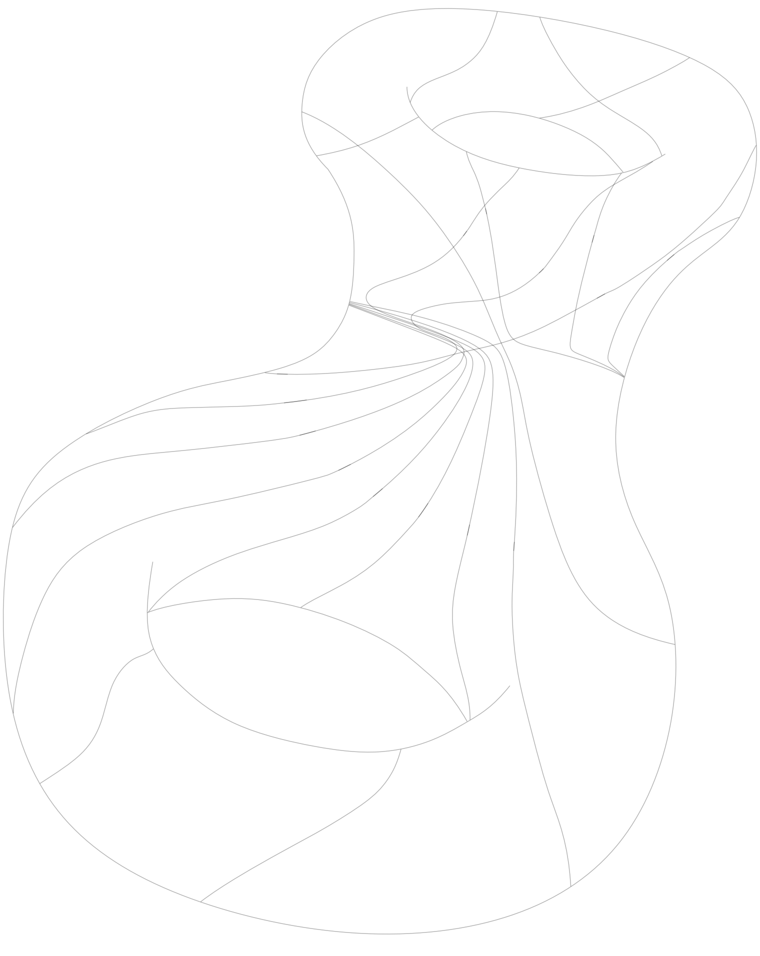
Abstract: We discuss suspension and contact flows to deduce the corresponding dichotomy for Anosov stable Hamiltonian structures in dimension three. Afterwards, we review the surgery proposed by Foulon and Hasselblatt to verify that Dehn surgery on flows can be refined to preserve stability of Anosov stable Hamiltonian structures. As a consequence, we obtain non-algebraic contact Anosov flows in dimension three. Furthermore, we expand on Foulon and Hasselblatt's surgery and show how to construct non-algebraic virtually contact Anosov Hamiltonian structures that are not of contact type. To obtain flows suitable for the latter type of surgery, we independently develop the rich theory of magnetic flows, including a thorough discussion of Mañé's critical values and associated results.



Introduction

An old important problem with many applications in both applied and pure mathematics is the study of (time-dependent) differential equations. Their solutions give rise to flows, which is the starting point for (continuous-time) dynamical systems. The approaches and results found in the theory of differential equations depend heavily on the type of the equation, which may range from elliptic to parabolic to hyperbolic. The latter is the kind we are interested in here. More specifically, we will mostly study Anosov flows, which are flows that admit hyperbolic behavior on the entire underlying manifold. One of the many nice concepts in the theory of Anosov flows is the interplay of dynamical systems and geometry. Namely, many properties of these flows are linked to the structure of the underlying manifold and vice versa. A key example of this are geodesic flows on Riemannian manifolds. For instance, these are Anosov if the manifold has strictly negative curvature. Geodesic flows model the evolution of a particle under the principle of least action. Starting from this, one may also introduce a distortion modeling a magnetic field. Then the resulting magnetic flow models the evolution of an electrically charged particle. While geodesic flows have been studied for centuries and are well-understood by now, the study of magnetic flows is only a few decades old and is still subject of active research. Obviously, a magnetic flow depends on both the Riemannian metric and the magnetic field. Various constellations of these two quantities relative to each other may equip the magnetic flow with properties like being Anosov or being contact. Let us explore a taste of this. Depending on the energy level of the charged particle, the magnetic field has a weaker or stronger influence. As the energy level increases, the magnetic flow approaches the geodesic flow. For negative curvature, the latter is Anosov and, since Anosov flows are structurally stable, the magnetic flow must be Anosov for large energy levels. Generally, one looses the Anosov property when considering a small charge, so there must be a specific energy level at which we pass from having an Anosov magnetic flow to having a non-Anosov one. This gives rise to the notion of critical values encompassing exactly this information. Surprisingly, these critical values also carry information about whether a magnetic flow has the contact or virtually contact property. Thus, we find another correlation between purely dynamical information and geometric data.

In a broader picture, the geometric consequences are even more noticeable, particularly in dimension three. A prominent indicator is Ghys' theorem, which states that any Anosov flow on a 3-manifold that is a circle bundle is orbit equivalent to a geodesic flow. On the other end of the spectrum, Plante proved that if the underlying manifold has a solvable fundamental group, then the flow is a suspension (which, morally, is a diffeomorphism artificially turned into a flow). Both of these examples fit into the wider class of Anosov Hamiltonian structures. These live at the intersection of dynamical systems and the symplectic/contact subbranch of geometry. An important notion that enters from the symplectic geometric side is stability. Stability of a hypersurface in a symplectic manifold refers to the existence of a tubular neighborhood that behaves nicely with respect to the ambient symplectic structure. This, in turn, is linked to the existence of a flow that realizes the tubular neighborhood by pushing the hypersurface in forward and backward time. It is such a flow we are interested in, i.e. a Reeb flow of an Anosov stable Hamiltonian structure. Whereas this class is much richer in higher dimensions, it is rather tight in dimension three. There is a dichotomy of the form contact versus suspension structures, and the geometry of the underlying manifold alone (almost) decides in which class the flow lives.

The subclass provided by contact Anosov flows particularly gives rise to interesting questions. For instance, in light of Ghys' theorem mentioned above, are there contact Anosov flows that are not equivalent to geodesic flows? The answer is "yes" but these flows are usually quite exotic and do not really arise naturally. Most counter-examples are constructed by performing Dehn surgery that preserves the Anosov property. Most of the times though, such a Dehn surgery breaks the contact property and one needs some additional care to obtain new Anosov flows that are still contact.

The preceding discussion emphasized contact flows, but there is a weaker notion one might be interested in. Namely, we will also study virtually contact Anosov Hamiltonian structures. Basically, this means that some lift of the structure is contact but with some additional boundedness assumptions. We can ask ourselves whether the Dehn surgery can be adapted so that it preserves the virtual contact property. In general, it seems like there are no decent starting points to tackle this problem. It is now that magnetic flows enter back into the game. These will provide flows that are suitable for such an adapted surgery. The upshot is that they are quite similar to geodesic flows which enables us to perform some explicit calculations. On the other hand, introducing a magnetic field is sufficient to break the contact property of geodesic flows in some specific settings. These two properties conveniently enable us to adapt the surgery to preserve the virtual contact property. We are not aware of any reference where such structures have been shown to exist and we believe that the construction of these virtually contact Anosov Hamiltonian structures is the highlight of this thesis.

Outline

Let us fix some standing assumptions, which we do not mention each time they are used. All manifolds are assumed to be smooth, connected, oriented, and closed. Furthermore, if we are dealing with a contact structure, then we always assume it is co-orientable, i.e. that the contact structure can be realized as the kernel of a globally defined smooth 1-form. Throughout, $\phi_t \colon M \to M$ denotes a smooth flow, whose infinitesimal generator we call F. We assume that any flow does not have fixed points.

The reader should be familiar with some basic notions from algebraic topology, symplectic, and contact geometry. Knowing some basic and intermediate results about hyperbolic flows can be useful, but we recall all the results we need in the first four sections of chapter one (though, we do not provide proofs for all of them). After that, we proceed with an elaborate discussion of suspension and contact flows (chapters 1.5 and 1.6). These are the set-up for the classification of Anosov stable Hamiltonian structures in dimension three (chapter 1.7).

In chapter two, we introduce magnetic flows on surfaces and begin by reviewing the special case in which the Riemannian metric has constant curvature and the magnetic field is a constant multiple of the associated area form (chapters 2.1, 2.2, and 2.3). Revisiting our interest in contact flows, we deduce that, in most cases, this is too strong of a property to satisfy for magnetic flows (chapter 2.4). This section is also the first step towards building suitable flows for the refined surgery later. Afterwards, we start tackling magnetic flows from a different point of view by introducing Lagrangians and the Legendre transform (chapter 2.5). This enables us to develop the theory surrounding Mañé's critical values, which we do in the proceeding section (chapter 2.6). We finish chapter two by exploring the interplay of magnetic dynamics and geometry, which is governed by the relation of the energy level relative to the previously defined critical values (chapter 2.7).

The final chapter is concerned with Dehn surgery giving rise to new exotic flows. We first show how Dehn surgery can preserve the Anosov property of a flow (chapter 3.1). Secondly, we exhibit exotic flows by producing Anosov flows that do not fit into the category of algebraic ones (chapter 3.2). Finally, we also want the surgery to preserve some geometric structure. More precisely, we look at Hamiltonian structures under surgery as well as what happens to stability (chapter 3.3). We do this based on work by Foulon and Hasselblatt. The thesis concludes with a construction of a certain class of new Hamiltonian structures of which there is no record in the literature that the author is aware of (chapter 3.4). More precisely, we prove the existence of Hamiltonian structures that are non-algebraic, Anosov, and of virtual contact type, but that do not arise from contact structures.

References

Here, we provide an overview of some useful references. To become familiar with hyperbolic flows, the reader may consult [FH18] and [KH95]. Some more advanced results on hyperbolic flows that we will use or review can be found in [HK90] and [Pla72]. Some specific results we mention are taken from [Bal95], [BFL92], [Ghy84], [Mat13], [Pla81], and [Tom70]. For an introduction to magnetic flows, we refer to [CFP10b], [CI99], and [MP11]. More advanced literature includes [BP02], [CIPP98], [Con06], [DP05], [DPSU07], [Gou97], [Mer10], [Mer16], [Pat97], [Pat06], and [PP97]. More generally, (stable) Hamiltonian structures are covered in [CFP10a], [CM05], [HT09], and [MP10]. For understanding the Dehn surgery for Anosov flows, some useful references are [FH13], [Goo83], and [HT80]. Lastly, the background material on geometry and topology can be found in the usual references. For instance, [Hat02] for algebraic topology, [CdS08] for symplectic geometry, [Gei08] for contact geometry, and [Bre93], [CLN85], [Kna96], [Lee13], [Sco83], and [ST67] as general literature on manifold theory.

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I am truly grateful for my supervisor Will Merry. He invested a lot of time to answer numerous questions and respectfully discuss the development of this thesis. His lectures and our briefings helped shape my mathematical thinking as well as my writing. Even more so, I am most grateful that he provided guidance throughout my studies even for matters unrelated to my (BSc and MSc) theses. I learned a lot from him and his advice. For that reason, I like to think of him not only as a supervisor but also as a mentor. He will always remain an inspiration for me.

Contents

1 Anosov Flows

1.1 Hyperbolicity

We begin by recalling a few notions concerning relations between flows. Two flows $\phi_t\colon M \to M$ and $\psi_t\colon N \to N$ are C^k*-conjugate* if there exists a C^k-diffeomorphism $h\colon M \to N$ with $h \circ \phi_t(x) = \psi_t \circ h(x)$. Conjugacies determine the topological (or C^k-regular) class of a flow since any topological (or smooth) property is preserved by C^0- (or C^k-)conjugacies. The next relation is weaker in the sense that it does not necessarily preserve topological properties. If $N = M$ and if the orbits of ϕ_t and ψ_t are the same (as subspaces of M), then ψ_t is said to be a *time-change* of ϕ_t. In this case, there is a family of continuous functions $\alpha_t\colon N \to N$ with $\psi_t(x) = \phi_{\alpha_t(x)}(x)$. Since ψ_t is a flow, this family must satisfy $\alpha_0 = 0$ and

$$\alpha_{s+t}(x) = \alpha_s(\psi_t(x)) + \alpha_t(x). \tag{1}$$

We call such a family α a *cocycle*. The time-change is said to be C^k if the cocycle is. Note that the infinitesimal generator of a time-change of ϕ_t is a multiple of F, namely multiplied by $\frac{\partial}{\partial t}\big|_{t=0}\alpha_t(x)$. We can also consider a hybrid type of relation between two flows: ϕ_t and ψ_t are C^k*-orbit equivalent* if there exists a C^k-diffeomorphism $h\colon M \to N$ carrying orbits of ϕ_t to orbits of ψ_t. In other words, the conjugate flow $h \circ \phi_t \circ h^{-1}$ is a time-change of ψ_t. We may assume that the time-change is smooth if we are willing to replace the conjugacy ([FH18, Prop. 1.3.20]. When we simply write that two flows are conjugate, time-changes, or orbit equivalent, then we refer to the weakest case $k = 0$. Since an orbit-equivalence contains a time-change, it also does not necessarily preserve all topological properties of a flow. However, note that all three types of relations preserve transitivity, where we recall that a flow is *transitive* if it admits a dense orbit.

For us, one of the most important properties a flow can have is hyperbolicity. Even more so, we are mainly interested in Anosov flows. Recall that a flow is *Anosov* if M is a *hyperbolic set* for the flow, meaning that there exists a ϕ_t-invariant splitting $TM = \mathbb{R}F \oplus E^s \oplus E^u$ in the sense that $(d\phi_t)_x(E^s_x) = E^s_{\phi_t(x)}$ (and similarly for E^u) that, moreover, satisfies the following: for some (any) Riemannian metric there exist constants $C \geq 1$ and $\mu \in (0,1)$ such that for any $x \in M$ and any $t \geq 0$ we have

$$||(d\phi_t)_x(v)|| \leq C\mu^t||v||, \text{ for all } v \in E^s_x,$$
$$||(d\phi_{-t})_x(v)|| \leq C\mu^t||v||, \text{ for all } v \in E^u_x.$$

That this property is independent of the Riemannian metric is due to compactness (a different choice of metric merely changes the constant C). A compact ϕ_t-invariant subset of M is *hyperbolic* if the restriction of the flow to this set is Anosov. We call $\mathbb{R}F$, E^s, and E^u the *center, stable*, and *unstable subbundles* and sometimes also write E^c for $\mathbb{R}F$. Note that, by invariance of the splitting, we also always have the reverse inequalities

$$||(d\phi_{-t})_x(v)|| \geq \frac{1}{C}\mu^{-t}||v||, \text{ for all } v \in E^s_x,$$
$$||(d\phi_t)_x(v)|| \geq \frac{1}{C}\mu^{-t}||v||, \text{ for all } v \in E^u_x.$$

By compactness of the Grassmanians, the splitting is always continuous and, hence, the dimension of each subbundle is constant. The splitting is even Hölder-continuous ([Bal95, page 81] or [FH18, Thrm. 8.3.1]) but, in general, does not inherit more regularity than that ([FH18, Prop. 8.4.7]). Clearly, the Anosov property of a flow is preserved under C^1-conjugacies. With some work, we can verify that the Anosov property is also preserved under C^1-time-changes:

Proposition 1.1 (Invariance under Time-Changes)**.** *Suppose ϕ_t is Anosov and ψ_t is a C^1-time-change of ϕ_t. Then ψ_t is Anosov, as well.*

Proof. Let $\alpha_t(x)$ denote the cocycle corresponding to the time-change. Abbreviate $f(x) = \frac{\partial}{\partial t}\big|_{t=0}\alpha_t(x)$ and write $\tilde{F} = fF$ for the infinitesimal generator of ψ_t. Then the differential of the latter is

$$(d\psi_t)_x(v) = (d\alpha_t)_x(v)F(\psi_t(x)) + (d\phi_{\alpha_t(x)})_x(v). \tag{2}$$

We want to construct the invariant splitting by hand. To this end, consider the subspaces $\tilde{E}^s_x$ given by $\{v + s(x,v)\tilde{F}(x) \mid v \in E^s_x\}$, where $s\colon TM \to \mathbb{R}$ needs to be specified. The differential of ψ_t acts on vectors in $\tilde{E}^s_x$ by

$$\begin{aligned} d\psi_t\left(v + s(x,v)\tilde{F}(x)\right) &= (d\alpha_t)_x(v)F(\psi_t(x)) + (d\phi_{\alpha_t(x)})_x(v) + s(x,v)\tilde{F}(\psi_t(x)) \\ &= (d\phi_{\alpha_t(x)})_x(v) + ((d\alpha_t)_x(v) + s(x,v)f(\psi_t(x)))\, F(\psi_t(x)). \end{aligned}$$

In order for the subbundle $\tilde{E}^s$ to be ψ_t-invariant, we need that this vector equals a vector of the form $w + s(\psi_t(x), w)\tilde{F}(\psi_t(x))$ with $w \in E^s_{\psi_t(x)}$. Thus, we need the function s to satisfy

$$(d\alpha_t)_x(v) + s(x,v)f(\psi_t(x)) = s\left(\psi_t(x), (d\phi_{\alpha_t(x)})_x(v)\right) f(\psi_t(x))$$

for all $x \in M$ and all $v \in E^s_x$. If we can find such a function s that is also linear in the second argument, then ψ_t fulfills the exponential decay on $\tilde{E}^s$ for

$$d\psi_t\left(v + s(x,v)\tilde{F}(x)\right) = (d\phi_{\alpha_t(x)})_x(v) + s\left(\psi_t(x), (d\phi_{\alpha_t(x)})_x(v)\right) f(\psi_t(x))F(\psi_t(x)) \tag{3}$$

inherits the exponential decay from $d\phi_{\alpha_t(x)}$ on E^s. The time-change does not disrupt the decay property since $\alpha_t(x)$ can be uniformly bounded from below by kt for some constant $k > 0$, by the pseudo-linearity of α with respect to time and by compactness of M. More precisely, the restriction of α to $M \times [1,2]$ takes values in a bounded domain $[\epsilon, R]$ for some $0 < \epsilon < R$, by compactness. Take k smaller than $\epsilon/2$. Then $\alpha_t(x) > kt$ holds on $M \times [1,2]$. For any larger $t > 2$, we get the same bound by the cocycle property. Fix any Riemannian metric on M. Using compactness once more and linearity of s in the second argument, we can pick a bound K of the operator norms of $s(x,\cdot)$, $x \in M$. Lastly, let K' denote a bound on $||\tilde{F}||$. Then equation (3) implies

$$\begin{aligned} ||d\psi_t\left(v + s(x,v)\tilde{F}(x)\right)|| &\leq (1 + KK')||(d\phi_{\alpha_t(x)})_x(v)|| \\ &\leq (1+KK')C\mu^{\alpha_t(x)}||v|| \leq (1+KK')C\mu^{kt}||v|| \end{aligned}$$

for all $t \geq 1$, where C and μ are the constants from the hyperbolicity definition for ϕ. By taking a larger constant, we can accommodate for values $t < 1$ and conclude that ψ satisfies the desired decay property on $\tilde{E}^s$. Hence, we only need to worry about invariance. Note that differentiating the cocycle equation (1) of α yields

$$(d\alpha_{r+t})_x = (d\alpha_t)_x + (d\alpha_r)_{\psi_t(x)} \circ (d\psi_t)_x.$$

In particular, using equation (2), we obtain

$$(d\alpha_r)_{\psi_t(x)} \circ \left(d\phi_{\alpha_t(x)}\right)_x(v) = (d\alpha_{r+t})_x(v) - (d\alpha_t)_x(v) - (d\alpha_t)_x(v)\cdot(d\alpha_r)_{\psi_t(x)}(F(\psi_t(x))). \tag{4}$$

Moreover, since

$$\tilde{F}(\psi_r(x)) = (d\psi_r)_x(\tilde{F}(x)) = (d\alpha_r)_x(\tilde{F}(x))F(\psi_r(x)) + (d\phi_{\alpha_r(x)})_x(\tilde{F}(x))$$

is equivalent to

$$f(\psi_r(x)) = f(x)\left(1 + (d\alpha_r)_x(F(x)\right),$$

we can conclude that

$$(d\alpha_r)_{\psi_t(x)}(F(\psi_t(x))) = \frac{f(\psi_{r+t}(x))}{f(\psi_t(x))} - 1.$$

Combining this equation with equation (4), we obtain

$$(d\alpha_r)_{\psi_t(x)} \circ \left(d\phi_{\alpha_t(x)}\right)_x (v) = (d\alpha_{r+t})_x(v) + \frac{f(\psi_{r+t}(x))}{f(\psi_t(x))}. \tag{5}$$

Having established this useful formula, we can turn to the quest of finding a suitable function s. Define an auxiliary function by

$$S(x, v) = \frac{\partial}{\partial r}|_{r=0} \frac{1}{f(\psi_r(x))}(d\alpha_r)_x(v).$$

Then this function behaves nicely when we plug in the input that s needs to take:

$$\begin{aligned} S\left(\psi_t(x), (d\phi_{\alpha_t(x)})_x(v)\right) &= \frac{\partial}{\partial r}\bigg|_{r=0} \frac{1}{f(\psi_{r+t}(x))}(d\alpha_r)_{\psi_t(x)}\left((d\phi_{\alpha_t(x)})_x(v)\right) \\ &\overset{(5)}{=} \frac{\partial}{\partial r}\bigg|_{r=0} \frac{1}{f(\psi_{r+t}(x))}\left((d\alpha_{r+t})_x(v) + \frac{f(\psi_{r+t}(x))}{f(\psi_t(x))}\right) \\ &= \frac{\partial}{\partial r}\bigg|_{r=0} \frac{1}{f(\psi_{r+t}(x))}(d\alpha_{r+t})_x(v) \\ &= \frac{\partial}{\partial r}\bigg|_{r=t} \frac{1}{f(\psi_r(x))}(d\alpha_r)_x(v). \end{aligned}$$

Next, consider the function

$$s(x, v) = -\int_0^\infty S\left(\psi_s(x), (d\phi_{\alpha_s(x)})_x(v)\right) ds.$$

The integral defining $s(x, v)$ exists because of the exponential decay of the second input of $S(x, v)$ coupled with linearity of S in the second argument. Further, s is exactly the function we were looking for as the following calculation shows:

$$\begin{aligned} s\left(\psi_t(x), (d\phi_{\alpha_t(x)})_x(v)\right) &= -\int_0^\infty S\left(\psi_{s+t}(x), (d\phi_{\alpha_{s+t}(x)})_x(v)\right) ds \\ &= -\int_t^\infty S\left(\psi_s(x), (d\phi_{\alpha_s(x)})_x(v)\right) ds \\ &= s(x, v) + \int_0^t \left(\frac{\partial}{\partial r}\bigg|_{r=s} \frac{1}{f(\psi_r(x))}(d\alpha_r)_x(v)\right) ds \\ &= s(x, v) + \frac{1}{f(\psi_t(x))}(d\alpha_t)_x(v). \end{aligned}$$

This finishes the construction of an invariant stable subbundle for ψ_t. An invariant unstable subbundle is obtained by analogue considerations. □

However, more than often it is not feasible to construct the splitting by hand as in the previous proof. To this end, it is convenient to introduce a more abstract and more flexible criterion to establish hyperbolicity. Such is provided by the Cone Criterion.

1.2 The Cone Criterion

Let $\epsilon \in (0,1)$. Given a split normed vector space $V = V_1 \oplus V_2$, whose elements we write as $v = v_1 + v_2$, we define the ϵ-cone by $C_\epsilon(V_1, V_2) = \{v \in V \mid ||v_2|| < \epsilon ||v_1||\}$.

Proposition 1.2 (Cone Criterion)**.** *A compact ϕ_t-invariant set $\Lambda \subset M$ is hyperbolic if and only if there exists a splitting $TM|_\Lambda = E^c \oplus S \oplus U$ and constants $C \geq 1$, $\epsilon, \mu \in (0,1)$ such that for all $x \in \Lambda$ and $t > 0$ we have the following (where we abbreviate $SC = E^c \oplus S$ and $UC = E^c \oplus U$):*

$$(d\phi_t)_x \left(\overline{C_\epsilon(U_x, SC_x)}\right) \subset C_\epsilon(U_{\phi_t(x)}, SC_{\phi_t(x)}),$$
$$(d\phi_{-t})_x \left(\overline{C_\epsilon(S_x, UC_x)}\right) \subset C_\epsilon(S_{\phi_{-t}(x)}, UC_{\phi_{-t}(x)}),$$

and in each cone

$$||(d\phi_t)_x(v)|| \geq \frac{1}{C}\mu^{-t}||v||, \ \textit{for } v \in C_\epsilon(U_x, SC_x),$$
$$||(d\phi_{-t})_x(v)|| \geq \frac{1}{C}\mu^{-t}||v||, \ \textit{for } v \in C_\epsilon(S_x, UC_x).$$

Proof. The "only if" direction is obvious since we are given an invariant splitting with the desired growth properties and compactness enables us to find small invariant cone neighborhoods in which the inequalities remain valid. Now suppose that we are given the setup described in the statement. We do not need to worry much about the growth conditions since these are already satisfied in the cones, but we need a splitting that is invariant. To this end, let us check that the following subsets qualify:

$$E^u_x = \bigcap_{t>0} (d\phi_t)_{\phi_{-t}(x)} \left(\overline{C_\epsilon(U_{\phi_{-t}(x)}, SC_{\phi_{-t}(x)})}\right),$$
$$E^s_x = \bigcap_{t>0} (d\phi_{-t})_{\phi_t(x)} \left(\overline{C_\epsilon(S_{\phi_t(x)}, UC_{\phi_t(x)})}\right).$$

Now abbreviate $S(r) = (d\phi_{-r})_{\phi_r(x)}(S_{\phi_r(x)})$. This is a linear space inside the cone $C_\epsilon(S_x, UC_x)$ so that for $t < r$ we have

$$(d\phi_t)_x(S(r)) \subset C_\epsilon(S_{\phi_t(x)}, UC_{\phi_t(x)}).$$

In particular,

$$S(r) \subset \bigcap_{t<r} (d\phi_{-t})_{\phi_t(x)} \left(\overline{C_\epsilon(S_{\phi_t(x)}, UC_{\phi_t(x)})}\right).$$

By compactness of the Grassmanians, we can consider an accumulation point $S(\infty)$ of $S(r)$. The last inclusion implies that this linear space $S(\infty)$ will be contained inside E^s_x. Moreover, $S(\infty)$ has the same dimension as S_x. Applying the same argument to E^u_x and counting dimensions, we find a splitting $T_xM = E^c_x \oplus S(\infty) \oplus U(\infty)$. If we can show that E^s_x is also contained in $S(\infty)$ (and $E^u_x \subset U(\infty)$), then this is exactly the desired hyperbolic splitting. Thus, suppose we are given a vector $v \in E^s_x$. Write this vector as $v^s + v^u$ with $v^s \in S(\infty)$ and $v^u \in E^c_x \oplus U(\infty)$. Clearly, the center space E^c_x does not belong to E^s_x so that, in fact, $v^u \in U(\infty) \subset E^u_x$. Then we can use the exponential growth to calculate

$$\begin{aligned} ||v^u|| &= ||(d\phi_{-t})_{\phi_t(x)} \circ (d\phi_t)_x(v^u)|| \leq C\mu^t ||(d\phi_t)_x(v^u)|| \\ &\leq C\mu^t \big(||(d\phi_t)_x(v)|| + ||(d\phi_t)_x(v^s)||\big) \leq C^2\mu^{2t}\big(||v|| + ||v^s||\big) \longrightarrow 0. \end{aligned}$$

□

Remark 1.3. *Examining the proof reveals that we never needed the cone structure. Indeed, all we need is a collection of subsets $C_x^s, C_x^u \subset TM$ on which the growth conditions are valid, whose forward (respectively backward) iterates define a strictly decreasing sequence of subsets, and which contain the splitting components S_x and U_x, respectively.*

As an application of the cone criterion, we may deduce the persistence of hyperbolicity. Namely, if Λ is a hyperbolic set for ϕ_t, then there exists a small neighborhood U of Λ such that for any sufficiently small C^1-perturbation ψ_t of ϕ_t the intersection $\bigcap_{t\in\mathbb{R}} \psi_t(\overline{U})$ is a hyperbolic set for ψ_t. Beware that the ψ_t-invariant set $\bigcap_{t\in\mathbb{R}} \psi_t(\overline{U})$ might be empty. One particular instance, in which we can assure that it is not empty, is the case $\Lambda = M$. This recovers the structural stability of Anosov flows.
For later use, we would also like to present a variant of the Cone Criterion. To state it, we introduce the following notion: A *Lorentz metric* on M is a collection g_x, $x \in M$, of non-degenerate bilinear forms on T_xM of signature $(m-1,1)$ that depend continuously on the base-point, where m denotes the dimension of M. We call the associated quadratic form $Q(v) = g_x(v,v)$, $v \in T_xM$, a *quadratic Lorentz form.* Given such a quadratic Lorentz form, we can define its positive cone at $x \in M$ as $C_x = \{v \in T_xM \mid Q_x(v) > 0\}$.

Proposition 1.4 (Cone Criterion, variant)**.** *If M has dimension three, ϕ_t is Anosov if and only if there are two quadratic Lorentz forms Q^s and Q^u with positive cones C^s and C^u, respectively, and constants $C \geq 1, \mu \in (0,1), c, T > 0$ such that for all $x \in M$*

(1.1) *for all $t > T$ and all $v \in C_x^s \setminus \{0\}$:* $Q^s_{\phi_{-t}(x)}((d\phi_{-t})_x)(v)) \geq \dfrac{1}{C}\mu^{-t} Q_x^s(v)$,

(1.2) *for all $t > T$ and all $v \in C_x^u \setminus \{0\}$:* $Q^u_{\phi_t(x)}((d\phi_t)_x)(v)) \geq \dfrac{1}{C}\mu^{-t} Q_x^u(v)$,

(2) *the intersection $C_x^s \cap C_x^u$ is empty,*

(3) $Q_x^s(F) \equiv -c \equiv Q_x^u(F)$,

(4.1) $(d\phi_{-T})_x \left(\overline{C_x^s} \setminus \{0\}\right) \subset C^s_{\phi_{-T}(x)}$,

(4.2) $(d\phi_T)_x \left(\overline{C_x^u} \setminus \{0\}\right) \subset C^u_{\phi_T(x)}$.

Proof. If we are given an Anosov flow, then such cones exist and, furthermore, the cones retroactively describe suitable Lorentz metrics. Let us prove the reverse implication. We will consider the projectivization PTM of the tangent bundle, i.e. collapse 1-dimensional linear subspaces to points. Because the Lorentz metrics forming Q^s and Q^u have positive signature 2 and the cones do not intersect, C^s and C^u become (filled-in) ellipses $\mathcal{E}^{\pm}$ in the projectivization. For a more convenient notation, let us write

$$\mathcal{E}_x^s(t) = P\left((d\phi_{-t})_{\phi_t(x)}\left(C^s_{\phi_t(x)}\right)\right) \subset PT_xM.$$

We claim that these ellipses $\mathcal{E}_x^s(t)$ collapse to a point as $t \to \infty$, which we prove further below. More precisely, we claim that the intersection $\mathcal{E}_x^s(\infty) = \bigcap_{t>T} \mathcal{E}_x^s(t)$ consists of a single point. Of course, we can do an analogue argument to get a singleton $\mathcal{E}_x^u(\infty)$. By property (2) in the assumptions, the ellipses $\mathcal{E}_x^s(t)$ and $\mathcal{E}_x^u(t)$ are disjoint for all $x \in M$ and all times t so that $\mathcal{E}_x^s(\infty)$ and $\mathcal{E}_x^u(\infty)$ are also disjoint for all $x \in M$. Moreover, by the third assumption, the vector field F does not lie in the cones and, hence, does not define the same point in PTM as $\mathcal{E}_x^s(\infty)$ or $\mathcal{E}_x^u(\infty)$. Now let us leave the projectivization and go back to the tangent bundle. The points $\mathcal{E}_x^s(\infty)$ and $\mathcal{E}_x^u(\infty)$ in PTM define two 1-dimensional subspaces E^s and E^u in TM whose intersection with each other and with $\mathbb{R}F$ is zero. These subbundles E^s and E^u are invariant because they are exactly

$$E_x^s = \bigcap_{t>T} (d\phi_{-t})_{\phi_t(x)}\left(\overline{C^s_{\phi_t(x)}}\right),$$

and similarly for E^u_x. For each $x \in M$, the set of vectors $v \in E^s_x, E^u_x$ with $Q^s_x(v) = 1$ or $Q^u_x(v) = 1$ consists of precisely two elements, so we can pick a Riemannian metric on M whose unit sphere intersects E^s_x, E^u_x in exactly these two points. Then the first hypothesis asserts the growth condition for $t > T$,

$$\frac{1}{C}\mu^{-t}||v||^2 = \frac{1}{C}\mu^{-t}||v||^2 \cdot Q^s_x\left(\frac{v}{||v||}\right) = \frac{1}{C}\mu^{-t}Q^s_x(v) \le Q^s_x\left((d\phi_{-t})_x(v)\right) = ||(d\phi_{-t})_x(v)||^2,$$

where $v \in E^s_x$, and similarly for E^u_x. By changing C we can also accommodate for values $t \le T$. It remains to prove the claim about the collapse of the ellipses. The fourth hypothesis asserts that $\overline{\mathcal{E}^s_x(T)} \subset \mathcal{E}^s_x$. In other words, if we fix any metric on PTM, then the map $P\left(d\phi_{-T}\right)_{\phi_T(x)} : \overline{\mathcal{E}^s_{\phi_T(x)}} \to \mathcal{E}^s_x$ induces a contraction of the diameter of the ellipses. By compactness of M, we can choose a uniform contraction rate r for all $x \in M$. Thus, for all $k \ge 0$

$$\text{diameter}\left(\overline{\mathcal{E}^s_x(kT)}\right) \le r^k \text{diameter}\left(\mathcal{E}^s_{\phi_{kT}(x)}\right).$$

Since the Lorentz metric varies continuously with the base-point, compactness implies that the diameters of $\mathcal{E}^s_x$ are uniformly bounded in x. Hence, the diameter of $\mathcal{E}^s_x(kT)$ tends to zero as k tends to infinity. This proves that $\bigcap_{k\ge0} \mathcal{E}^s_x(kT)$ consists of a single point. Even though the fourth property does not assert that the ellipses are nested for all times, the first property does exactly that for the interior of the ellipses. Therefore, $\bigcap_{k\ge0} \mathcal{E}^s_x(kT) = \bigcap_{t>T} \mathcal{E}^s_x(t)$, which finishes the proof of the claim. □

Let us note that the fact that we required the first hypothesis only for $t > T$ and the fourth hypothesis only for $t = T$ changed little in the proof but gives us much more flexibility when verifying the hypothesis. Conversely, if we start with an Anosov flow, then we can certainly satisfy the first and fourth hypothesis for all times $t > 0$.

Remark 1.5. *We stated the criterion for manifolds of dimension three. However, the criterion remains true in higher dimensions, only then the quadratic forms are not Lorentz forms but have signatures* $(m - \dim(E^s), \dim(E^s))$ *and* $(m - \dim(E^u), \dim(E^u))$*, respectively.*

1.3 Geodesic Flows

In this section, we will review one of the most prominent and most studied examples of flows, namely geodesic flows. Fix a Riemannian metric g on M so that there is a well-defined notion of geodesics. We include the unit-speed property in the definition of a geodesic. Denote by UM the unit tangent bundle of M. The *geodesic flow* on (M, g) is the flow defined by

$$\phi_t \colon UM \to UM,\ \phi_t(x, u) = (\gamma_u(t), \dot\gamma_u(t)),$$

where γ_u is the unique geodesic starting at x with velocity u. Its infinitesimal generator is

$$G(x, u) = (\partial_t|_{t=0}\gamma_u(t), \nabla_t|_{t=0}\dot\gamma_u(t)) = (u, 0),$$

where we use the splitting of the tangent bundle discussed in the appendix and ∇ denotes the Levi-Civita connection of the metric. Our goal is to use the Cone Criterion to establish that the geodesic flow is Anosov. The appropriate conditions for this to be true are as follows:

Theorem 1.6. *If the Riemannian manifold M has strictly negative sectional curvature, then the geodesic flow is Anosov.*

Recall that the sectional curvature of a 2-dimensional plane $S \subset T_xM$ is

$$K(S) = \frac{\langle R(u, v)u, v\rangle}{\langle u, u\rangle\langle v, v\rangle - \langle u, v\rangle^2},$$

where $u, v \in T_xM$ are any two vectors spanning S, and R is the Riemann curvature tensor. The latter is defined as

$$R(u,v)w = (\nabla_t \nabla_s Z - \nabla_s \nabla_t Z)(0,0),$$

where $u, v, w \in T_xM$, $\gamma\colon \mathbb{R}^2 \to M$, and Z is a vector field along γ with

$$\gamma(0,0) = x, \quad \partial_s \gamma(0,0) = u, \quad \partial_t \gamma(0,0) = v, \quad Z(0,0) = w.$$

To prove the theorem, we make use of the theory of Jacobi fields. A *Jacobi field* along a geodesic γ is a vector field Y along γ that is a solution to the *Jacobi equation*

$$\nabla_t \nabla_t Y + R(\dot\gamma, Y)\dot\gamma = 0. \tag{6}$$

A result from differential geometry states that Jacobi fields arise as perturbations of geodesics. Indeed, if $\gamma(t) = \exp_x(tv)$, for small t, and we look at a perturbation of γ,

$$Y(t) = \frac{\partial}{\partial s}\Big|_{s=0} \exp_{\alpha(s)}(tV(s)),$$

where $\alpha(s) \in M$ and $V(s) \in T_{\alpha(s)}M$ satisfy $\alpha(0) = x$ and $V(0) = v$, then Y solves the Jacobi equation. Indeed, abbreviate $\Gamma(s,t) = \exp_{\alpha(s)}(tV(s))$ so that $\nabla_t \partial_t \Gamma = 0$ because for fixed s the map $t \mapsto \Gamma(s,t)$ is a geodesic. Then

$$\nabla_t \nabla_t \partial_s \Gamma = \nabla_t \nabla_s \partial_t \Gamma - \nabla_s \nabla_t \partial_t \Gamma = R(\partial_s \Gamma, \partial_t \Gamma)\partial_t \Gamma.$$

Evaluating this equation at $s = 0$ yields the Jacobi equation. In fact, any Jacobi field has this form of a perturbation. To see this, suppose a Jacobi field Y_* along a geodesic γ is given. Locally, $\gamma(t) = \exp_x(tv)$, where $x = \gamma(0)$ and $v = \dot\gamma(0)$, and we can let $Y(t)$ be defined as above. If we pick α such that $\dot\alpha(0) = Y_*(0)$ and also pick $V(s)$ such that $\nabla_s V(0) = \nabla_t Y(0)$, then we get $Y(0) = \dot\alpha(0) = Y_*(0)$ as well as $\nabla_t Y(0) = \nabla_s V(0) = \nabla_t Y_*(0)$. Thus, Y and Y_* satisfy the same differential equation with the same initial conditions. By standard ODE theory, we can conclude that $Y \equiv Y_*$, at least for $||tv||$ smaller than the injectivity radius where Y is well-defined. Now let us turn back to the geodesic flow. For small t, we can write

$$\phi_t(x,u) = \left(\exp_x(tu), \partial_r|_{r=t} \exp_x(ru)\right).$$

Suppose Y_0 is a given vector in the tangent space $T_{(x,u)}UM$ of the unit tangent bundle. Take a path β in UM with $\dot\beta(0) = Y_0$ and write it as $\beta(s) = (\alpha(s), V(s))$. Then we can compute

$$\begin{aligned}(d\phi_t)_{(x,u)}(Y_0) &= \partial_s\Big|_{s=0} \phi_t \circ \beta(s) = \left(\partial_s\Big|_{s=0} \exp_{\alpha(s)}(tV(s)), \nabla_s\Big|_{s=0} \partial_r\Big|_{r=t} \exp_{\alpha(s)}(rV(s))\right) \\ &= (Y(t), \nabla_t Y(t)),\end{aligned}$$

where $Y(t)$ is, as above, the Jacobi field with initial conditions

$$(Y(0), \nabla_t Y(0)) = (\dot\alpha(0), \nabla_s V(0)) = \partial_s \beta(0) = Y_0.$$

Thus, we can express the differential of the geodesic flow as Jacobi fields.

Proof of theorem 1.6. We want to use the Cone Criterion. By compactness, it suffices to fulfill the growth properties in the cones in any norm, not necessarily the one induced by the metric. Let us define a new norm on each $T_{(x,u)}UM$ by

$$||(X_H, X_V)||_\delta = \sqrt{\langle X_H, X_H\rangle + \delta \langle X_V, X_V\rangle},$$

for some fixed $\delta > 0$ specified later. Consider the cones

$$C^{+\epsilon}_{(x,u)} = \left\{ (X_H, X_V) \in T_{(x,u)}UM \;\middle|\; \langle u, X_H\rangle = 0 = \langle u, X_V\rangle \text{ and } \langle X_H, X_V\rangle \geq \epsilon ||(X_H, X_V)||^2_\delta \right\},$$
$$C^{-\epsilon}_{(x,u)} = \left\{ (X_H, X_V) \in T_{(x,u)}UM \;\middle|\; \langle u, X_H\rangle = 0 = \langle u, X_V\rangle \text{ and } \langle X_H, X_V\rangle \leq -\epsilon ||(X_H, X_V)||^2_\delta \right\}.$$

These are not actual cones but we can show that they are "cones" in the sense of remark 1.3. We claim that these sets satisfy the hypothesis of the cone criterion for suitable ϵ and δ. Firstly, note that none of the cones intersects span$\langle G\rangle$. Furthermore, each cone contains an $(n-1)$-dimensional linear subspace of $T_{(x,u)}UM$: if $u, v_1, \ldots, v_{n-1}$ is an orthonormal basis of T_xM, then $(v_j, v_j) \in C^{+\epsilon}_{(x,u)}$ and $(-v_j, v_j) \in C^{-\epsilon}_{(x,u)}$ for every $1 \leq j \leq n-1$ if we can take $\epsilon(1+\delta) < 1$. We need to assert that there are suitable choices for δ and ϵ such that we get strict invariance of the cone field and the growth conditions are satisfied inside the cones. Given $(X_H, X_V) \in \overline{C^{+\epsilon}_{(x,u)}}$, the cone criterion requires $(d\phi_t)_{(x,u)}(X_H, X_V) \in C^{+\epsilon}_{\phi_t(x,u)}$. By the previous discussion, this reduces to showing $(Y(0), \nabla_t Y(0)) \in \overline{C^{+\epsilon}_{(x,u)}}$ implies $(Y(t), \nabla_t Y(t)) \in C^{+\epsilon}_{\phi_t(x,u)}$ for a Jacobi field Y. To this end, it suffices to prove that

$$\frac{\langle Y(t), \nabla_t Y(t)\rangle}{||(Y(t), \nabla_t Y(t))||^2_\delta} = \epsilon \implies \frac{d}{dt} \frac{\langle Y(t), \nabla_t Y(t)\rangle}{||(Y(t), \nabla_t Y(t))||^2_\delta} > 0.$$

To get some decent bounds, it is useful to observe that the Jacobi fields in question are orthogonal to the geodesic γ_u in the sense that both $\langle\dot\gamma, Y\rangle \equiv 0$ and $\langle\dot\gamma, \nabla_t Y\rangle \equiv 0$. Indeed, they are orthogonal to u at time 0 by definition of the cones and that they remain orthogonal for all times follows from

$$\frac{d}{dt}\langle\dot\gamma, \nabla_t Y\rangle = -\langle\dot\gamma, R(\dot\gamma, Y)\dot\gamma\rangle = 0 \quad \text{and} \quad \frac{d}{dt}\langle\dot\gamma, Y\rangle = \langle\dot\gamma, \nabla_t Y\rangle = 0.$$

By compactness, the sectional curvature is bounded from above by some $-k^2 < 0$. Together with orthogonality, this yields

$$\langle R(\dot\gamma, Y)\dot\gamma, Y\rangle \leq -k^2\langle Y, Y\rangle. \tag{7}$$

By compactness of UM, we may also bound

$$\langle R(\dot\gamma, Y)\dot\gamma, R(\dot\gamma, Y)\dot\gamma\rangle \leq \frac{1}{\kappa^2}\langle Y, Y\rangle \tag{8}$$

where $1/\kappa > 0$ bounds the operator norm of $Y \mapsto R(\dot\gamma, Y)\dot\gamma$. To simplify notation, let us abbreviate $\dot Y = \nabla_t Y$ and $RY = R(\dot\gamma, Y)\dot\gamma$. Note that if $\delta \leq 1/k^2$, then

$$\frac{\langle\dot Y, \dot Y\rangle + k^2\langle Y, Y\rangle}{||(Y, \dot Y)||^2_\delta} = k^2\left(\frac{\frac{\langle Y,Y\rangle}{\langle\dot Y,\dot Y\rangle} + \frac{1}{k^2}}{\frac{\langle Y,Y\rangle}{\langle\dot Y,\dot Y\rangle} + \delta}\right) = k^2\left(1 + \frac{\frac{1}{k^2} - \delta}{\frac{\langle Y,Y\rangle}{\langle\dot Y,\dot Y\rangle} + \delta}\right) \geq k^2 \tag{9}$$

Moreover, we have

$$\frac{\sqrt{\langle\dot Y, \dot Y\rangle\langle Y, Y\rangle}}{||(Y, \dot Y)||^2_\delta} \leq \frac{1}{2\sqrt{\delta}} \tag{10}$$

since $(\langle Y, Y\rangle - \delta\langle \dot{Y}, \dot{Y}\rangle)^2 \geq 0$. We can now compute using $\frac{\langle Y,\dot{Y}\rangle}{||(Y,\dot{Y})||_\delta^2} = \epsilon$ and the above bounds

$$\begin{aligned}
\frac{d}{dt}\frac{\langle Y,\dot{Y}\rangle}{||(Y,\dot{Y})||_\delta^2} &= \frac{\left(||(Y,\dot{Y})||_\delta^2\left(\langle \dot{Y},\dot{Y}\rangle + \langle Y,\ddot{Y}\rangle\right) - 2\langle Y,\dot{Y}\rangle\left(\langle Y,\dot{Y}\rangle + \delta\langle \dot{Y},\ddot{Y}\rangle\right)\right)}{||(Y,\dot{Y})||_\delta^4} \\
&\overset{(6)}{=} \frac{\langle \dot{Y},\dot{Y}\rangle - \langle Y, RY\rangle}{||(Y,\dot{Y})||_\delta^2} - 2\epsilon\left(\epsilon - \delta\frac{\langle \dot{Y}, RY\rangle}{||(Y,\dot{Y})||_\delta^2}\right) \\
\text{(Cauchy-Schwarz)}\quad &\overset{(7)}{\geq} \frac{\langle \dot{Y},\dot{Y}\rangle + k^2\langle Y,Y\rangle}{||(Y,\dot{Y})||_\delta^2} - 2\epsilon\left(\epsilon + \delta\frac{\sqrt{\langle \dot{Y},\dot{Y}\rangle\langle RY,RY\rangle}}{||(Y,\dot{Y})||_\delta^2}\right) \\
&\overset{(8),(9)}{\geq} k^2 - 2\epsilon\left(\epsilon + \frac{\delta}{\kappa}\frac{\sqrt{\langle \dot{Y},\dot{Y}\rangle\langle Y,Y\rangle}}{||(Y,\dot{Y})||_\delta^2}\right) \\
&\overset{(10)}{\geq} k^2 - 2\epsilon\left(\epsilon + \frac{\sqrt{\delta}}{2\kappa}\right) \overset{\delta\leq 1/k^2}{\geq} k^2 - 2\epsilon\left(\epsilon + \frac{1}{2\kappa k}\right) \\
&> 0 \quad \text{if} \quad \epsilon < \sqrt{\frac{k^2}{2} + \frac{1}{16\kappa^2 k^2}} - \frac{1}{4\kappa k}.
\end{aligned}$$

This establishes strict invariance of the cone field. To verify exponential growth, we calculate further with the same estimates as in the previous computation

$$\begin{aligned}
\frac{\frac{d}{dt}||(Y,\dot{Y})||_\delta^2}{||(Y,\dot{Y})||_\delta^2} &\overset{(6)}{=} 2\frac{\langle Y,\dot{Y}\rangle}{||(Y,\dot{Y})||_\delta^2} - 2\delta\frac{\langle \dot{Y}, RY\rangle}{||(Y,\dot{Y})||_\delta^2} \\
&\geq 2\epsilon - \frac{\sqrt{\delta}}{\kappa} > 0 \quad \text{if} \quad \delta < 4\epsilon^2\kappa^2.
\end{aligned}$$

This yields $||(Y,\dot{Y})||_\delta^2 \geq C\mu^t$, where $C = ||(Y(0),\dot{Y}(0))||_\delta^2$ and $\mu = \exp(2\epsilon - \sqrt{\delta}/\kappa)$. In particular, we can conclude that $|||d\phi_t||$ grows exponentially on $C^{+\epsilon}_{(x,u)}$. Note that we picked ϵ independently of δ, so by taking δ smaller than $\min\left(\frac{1}{\epsilon} - 1, 1/k^2, 4\epsilon^2\kappa^2\right)$ all the inequalities hold simultaneously. The computations for $C^{-\epsilon}_{(x,u)}$ are analogous. This finishes the proof. □

Geodesic flows have many more beautiful properties, but we will not discuss them here. They will appear again as examples of the more general class of contact flows. However, before we get to these, we first study a few more fundamental results associated with hyperbolic flows.

1.4 Invariant Foliations

In this section, we recall two more standard facts about hyperbolic flows. The first one is the existence of invariant foliations associated to the subbundles of the splitting. The second one is the Spectral Decomposition. Fix a Riemannian metric on M and let d be the associated distance function. The next theorem can be found as [FH18, Thrm. 6.1.1].

Theorem 1.7 (Local Stable and Unstable Manifold Theorem). *Suppose $\Lambda \subset M$ is a hyperbolic set for a flow ϕ_t. There exists a constant $\epsilon_0 > 0$ such that for all $\epsilon < \epsilon_0$ and all $x \in \Lambda$ the sets*

$$W^s_\epsilon(x) = \left\{ y \in M \;\middle|\; \forall t > 0\colon d(\phi_t(x)\phi_t(y)) < \epsilon \text{ and } d(\phi_t(x),\phi_t(y)) \to 0 \text{ as } t \to \infty \right\},$$
$$W^u_\epsilon(x) = \left\{ y \in M \;\middle|\; \forall t > 0\colon d(\phi_{-t}(x)\phi_{-t}(y)) < \epsilon \text{ and } d(\phi_{-t}(x),\phi_{-t}(y)) \to 0 \text{ as } t \to \infty \right\}$$

are embedded smooth manifolds that depend continuously on the base-point in the C^1-topology. Moreover, their tangent bundles are given by $T_xW^s_\epsilon(x) = E^s_x$ and $T_xW^u_\epsilon(x) = E^u_x$, and they are invariant in the sense that for $t > 0$ we have

$$\phi_t\left(W^s_\epsilon(x)\right) \subset W^s_\epsilon(\phi_t(x)) \quad \textit{and} \quad \phi_{-t}\left(W^u_\epsilon(x)\right) \subset W^u_\epsilon(\phi_{-t}(x)).$$

Furthermore, for some $C \geq 1$ and $\mu \in (0,1)$ we have for all $t > 0$

$$d(\phi_t(x), \phi_t(y)) < C\mu^t d(x,y), \ \textit{for all } y \in W^s_\epsilon(x),$$
$$d(\phi_{-t}(x), \phi_{-t}(y)) < C\mu^t d(x,y), \ \textit{for all } y \in W^u_\epsilon(x).$$

We call these manifolds the *local strong stable* and *local strong unstable manifolds.* In contrast, we define the *(global) strong (un)stable manifolds* as

$$W^s(x) = \bigcup_{t>0} \phi_{-t}\left(W^s_\epsilon(\phi_t(x))\right) = \left\{ y \in M \;\middle|\; d(\phi_t(x), \phi_t(y)) \to 0 \text{ as } t \to \infty \right\},$$
$$W^u(x) = \bigcup_{t>0} \phi_t\left(W^s_\epsilon(\phi_{-t}(x))\right) = \left\{ y \in M \;\middle|\; d(\phi_{-t}(x), \phi_{-t}(y)) \to 0 \text{ as } t \to \infty \right\}.$$

These are not embedded, but immersed manifolds. Note that $W^s(x) \cap W^s(y) \neq \emptyset$ implies $W^s(x) = W^s(y)$ by the triangle inequality and likewise for the unstable manifold. In particular, $T_yW^s(x) = E^s_y$ and $T_yW^u(x) = E^u_y$ for any points $x \in \Lambda$ and $y \in W^s(x), W^u(x)$. Lastly, as the name giving suggests, there are corresponding *(global) weak (un)stable manifolds* given by

$$W^{sc}(x) = \bigcup_{t \in \mathbb{R}} \phi_t\left(W^s(x)\right) \quad \text{and} \quad W^{uc}(x) = \bigcup_{t \in \mathbb{R}} \phi_t\left(W^u(x)\right).$$

For these we also have that $W^{sc}(x) \cap W^{sy}(x) \neq \emptyset$ implies $W^{sc}(x) = W^{sc}(y)$ as well as $T_yW^{sc}(x) = E^c_y \oplus E^s_y$ and likewise for the unstable case. Because these (un)stable manifolds cannot intersect each other, they give rise to foliations. We call the resulting foliations the *strong/weak (un)stable foliation.* The above theorem includes the statement that these foliations have smooth leaves. However, in direction transverse to the leaves, the regularity is usually diminished. From Hölder-continuity of the splitting components, we can deduce at most that the foliation is Hölder-continuous. However, something special happens in dimension three: the dimension alone is sufficient to enforce transverse C^1-regularity on the weak (un)stable foliation ([KH95, Cor. 19.1.11] or [FH18, Cor. 8.3.15]). Much more can be said about the regularity of the foliations, but we do not need an in-depth knowledge of that here. Let it be said that we will later investigate contact Anosov flows, which enforce smoothness on the bundle $E^s \oplus E^u$ and, hence, C^1-regularity on the strong (un)stable foliation.
We now turn to the Spectral Decomposition. Recall that a *basic set* for ϕ_t is an isolated hyperbolic set such that the restriction of the flow is transitive. The *non-wandering set* of ϕ_t is the set of all points for which an arbitrarily small neighborhood returns to itself in finite time. The following classical theorem even holds under weaker hypothesis but we only use it in the Anosov case; for a proof, see [FH18, Thrm. 5.2.22].

Theorem 1.8 (Spectral Decomposition)**.** *For an Anosov flow, the non-wandering set is the closure of the periodic points and is a finite disjoint union of basic sets. Furthermore, each basic set is the closure of an equivalence class of the following equivalence relation on the set of periodic points:*

$$p \sim q \iff W^{sc}(p) \cap W^{uc}(q) \neq \emptyset \ \ \textit{and} \ \ W^{sc}(q) \cap W^{uc}(p) \neq \emptyset$$

Corollary 1.9. *An Anosov flow is transitive if and only if its non-wandering set equals M.*

Next, we prove some results related to density of (un)stable manifolds and to identifying the Spectral Decomposition. These will come in handy when we later study volume-preserving flows.

Lemma 1.10. *Given any basic set Ω_0 and a periodic point $p \in \Omega_0$ of an Anosov flow ϕ_t, the weak (un)stable manifold $W^{sc}(p)$ ($W^{uc}(p)$) is dense in Ω_0.*

Proof. By considering ϕ_{-t}, it suffices to prove the statement for $W^{uc}(p)$. Furthermore, since the periodic points are dense in the non-wandering set, it suffices to show that $W^{uc}(p)$ accumulates on any periodic point $q \in \Omega_0$. By the characterization of the basic sets in the Spectral Decomposition, there is some point $z \in W^{sc}(q) \cap W^{uc}(p)$. Then some iterate $\phi_{t_1}(z)$ lies in $W^s(q) \cap W^{uc}(p)$. Therefore, since q is periodic, another iterate $\phi_{t_1+t_2}(z)$ is close to q and still in $W^{uc}(p)$. $\square$

From this, we can deduce the following result about when the Spectral Decomposition is trivial.

Lemma 1.11. *If the non-wandering set of an Anosov flow ϕ_t contains an open set, then it equals the entire manifold. In particular, the Spectral Decomposition is trivially $\{M\}$.*

Proof. Denote by Ω_0 a basic set in the Spectral Decomposition that contains an open set U. It suffices to show that Ω_0 is open. Given $p \in \Omega_0$, we first show that $W^{sc}(p)$ and $W^{uc}(p)$ are both contained in Ω_0 and are dense. Since periodic points are dense in the non-wandering set, there exists a periodic point $p_0 \in U$. Given $q \in W^{sc}(p_0)$, the orbit of q asymptotically approaches the orbit of p_0. In particular, the orbit of q hits the open set U at some point. Thus, $W^{sc}(p_0) \subset \bigcup_{t \le 0} \phi_t(U) \subset \Omega_0$ by invariance. Since Ω_0 is closed and since $W^{sc}(p_0)$ is dense in Ω_0 by the previous lemma, we find $\overline{W^{sc}(p_0)} = \Omega_0$. Hence, there is a sequence $(p_n)_n \subset W^{sc}(p_0)$ converging to p. Then also $\overline{W^{sc}(p)} = \Omega_0$ because $W^{sc}(p_n) = W^{sc}(p_0)$ and the weak stable manifolds depend continuously on the base-point. The entire argument also works for $W^{uc}(p)$. Note that at p the submanifolds $W^s(p)$ and $W^u(p)$ are transverse to each other as well as transverse to the flow direction. Thus, by their density, $\overline{W^{sc}(p)}$ and $\overline{W^{uc}(p)}$ "foliate" a small open neighborhood of p. This neighborhood is contained in Ω_0, which finishes the proof. $\square$

Let us note that the proof of lemma 1.11 contained the following upgrade of lemma 1.10:

Lemma 1.12. *If the non-wandering set is the entire manifold, then $W^{sc}(p)$ and $W^{uc}(p)$ are dense in M for any point $p \in M$.*

We obtained results about density of the weak (un)stable manifolds. When we ask for density of the strong (un)stable manifolds, we encounter another topological property of flows. Recall that a flow is *topologically mixing* if for any two open sets U and V there is some time $T > 0$ with $\phi_t(U) \cap V \neq \emptyset$ for all $t \ge T$. Clearly, topological mixing implies transitivity, but the reverse implication does not hold in general.

Proposition 1.13. *If ϕ_t is a transitive Anosov flow and if the strong stable and strong unstable manifold of every periodic point is dense, then ϕ_t is topologically mixing.*

Proof. Pick any metric on M and denote by $W^u_R(q)$ the ball of radius R around q inside the strong unstable manifold with respect to the induced metric on $W^u(q)$. Let p be a periodic point and $\epsilon > 0$. We make the following claim: there exists some large $R > 0$ so that for any point q on the orbit of p the set $W^u_R(q)$ is ϵ-dense in M. Suppose for contradiction that for all $n \in \mathbb{N}$ there is a point q_n on the orbit of p and a point $z_n \in M$ so that $W^u_n(q_n)$ and $B_\epsilon(z_n)$ are disjoint. By passing to a subsequence, we may assume $q_n \to q$ and $z_n \to z$ as $n \to \infty$. But then $W^u(q)$ and $B_\epsilon(z)$ are disjoint contradicting the hypothesis. To finish the proof, let U and V be two open sets and pick a periodic point $p \in U$ and some $\epsilon > 0$ so that $W^u_\epsilon(p) \subset U$. Such a periodic point exists by transitivity. Let $R > 0$ be given by the claim. Since we use the unstable manifold, $\phi_t(W^u_\epsilon(p))$ will contain $W^u_R(\phi_t(p))$ for large times, i.e. for $t \ge T$ and some large $T > 0$. Density of $W^u_R(\phi_t(p))$ from the claim implies $\phi_t(U) \cap V \neq \emptyset$. $\square$

The converse of this proposition is also true, which we will encounter in the next section.

1.5 Suspension Flows

We already introduced geodesic flows as a prominent example of Anosov flows. Another important class of examples is given by *suspension flows*. These are constructed as follows. Suppose $f\colon N \to N$ is a diffeomorphism of a manifold. Define the *suspension* (also called *mapping torus*) M_f of f as the quotient of $N \times \mathbb{R}$ by the equivalence relation $(x, s) \sim (f(x), s-1)$. Note that this is a fiber bundle over S^1. Then f induces a flow on its suspension via $\phi_t([x, s]) = [x, t+s]$. Assume now that f is Anosov, i.e. TM splits into the stable and unstable subbundles E^s and E^u. Then ϕ_t is an Anosov flow with the same stable and unstable subbundles. This can be seen readily after noting that the infinitesimal generator is simply the vector field $\frac{\partial}{\partial t}$ generated by the time coordinate and that $(d\phi_t)_{[x,s]} = ((df^{\lfloor s+t \rfloor})_x, \mathrm{id})$. In dimension three, there is a natural constraint on the manifolds admitting Anosov suspensions.

Proposition 1.14. *If a suspension flow is Anosov, then the suspension came from an Anosov diffeomorphism. In dimension three, the suspended manifold must have been a torus.*

Proof. Since a suspension flow clearly leaves the tangent bundles of the fibers invariant, the latter must coincide with the bundle $E^s \oplus E^u$. Thus, the Anosov splitting of the flow also defines an Anosov splitting for the time-1 map of the flow, which is exactly the suspended diffeomorphism. In dimension three, the suspended manifold is a closed oriented surface. Since the existence of an Anosov splitting implies the existence of a (continuous) nowhere vanishing vector field, this surface can only have been a torus. □

In the previous section, we discussed results related to density of (un)stable manifolds. On the contrary, suspensions are linked to non-density of these:

Proposition 1.15. *If ϕ_t is a transitive Anosov flow and if there exists a periodic point whose strong stable or strong unstable manifold is not dense, then ϕ_t is a (C^1-)suspension.*

For a proof of this result, we refer to [Pla72, Theorem 1.8]. As a consequence, we obtain a dichotomy of the form suspension versus mixing for transitive Anosov flows. Indeed, note that suspension flows can never be topologically mixing because, for instance, the sets $\phi_t(N \times (0, \frac{1}{2}))$ and $N \times (\frac{1}{2}, 1)$ are disjoint for all $t \in \mathbb{Z}$. Thus, combining propositions 1.13 and 1.15 yields:

Corollary 1.16. *A transitive Anosov flow is either topologically mixing or a suspension.*

We would like to develop another criterion to detect suspensions. In order to do so, we review some useful results about integrability of bundles and 1-forms. Usually, we consider smooth (C^∞) forms, but in this section we will allow continuous forms, as well. However, in the other chapters, forms are still assumed to be smooth unless explicitly stated otherwise.
For a continuous form, there is an adapted definition of an exterior derivative, which agrees with the standard notion in the smooth case. We are mainly interested in 1-forms. A 1-form λ admits an *exterior derivative* if there exists a 2-form μ such that $\int_{\partial B} \lambda = \int_B \mu$ for every C^1-immersed disk B with a piecewise C^1 boundary. If λ admits an exterior derivative, then it is unique and we denote it by $d\lambda$. We call a 1-form *closed* if its exterior derivative exists and is zero. A 1-form λ is said to be *integrable* if the bundle $\ker(\lambda)$ is integrable, i.e. if there exists a foliation $\mathcal{F}$ with $T\mathcal{F} = \ker(\lambda)$. The following is a standard result in the theory of ODE's, originally proved by Frobenius:

Theorem 1.17 (Frobenius' Theorem)**.** *A 1-form λ is integrable if and only if its exterior derivative exists and $\lambda \wedge d\lambda = 0$. In particular, any closed 1-form is integrable.*

Sketch of proof. We give an outline of a proof for the smooth case. The idea is to use flows induced by two linearly independent vector fields in $\ker(\lambda)$ to sweep out integral submanifolds. That these submanifolds really are integral to $\ker(\lambda)$ is ensured when the two flows commute, which in turn is ensured when the Lie bracket of the two vector fields vanishes. Observe that $\lambda \wedge d\lambda = 0$ if and only if $[X, Y] \in \ker(\lambda)$

for all $X, Y \in \ker(\lambda)$. Indeed, this follows readily from $d\lambda(X,Y) = \lambda([X,Y])$ for $X, Y \in \ker(\lambda)$. If λ is integrable and N is an integral submanifold, then surely $X, Y \in TN$ implies $[X,Y] \in TN \subset \ker(\lambda)$. Conversely, if the latter condition is satisfied, then one can construct two linearly independent vector fields spanning $\ker(\lambda)$ whose Lie bracket vanishes. For more details, we refer to [Har02, p. 123f.]. □

Given a closed 1-form λ, consider the homomorphism $H_1(M,\mathbb{Z}) \to \mathbb{R}$ that sends a homology class c to $\int_\sigma \lambda$, where σ is a smooth representative of c. This is well-defined because for any boundary $\delta\eta$

$$\int_{\sigma+\delta\eta} \lambda = \int_\sigma \lambda + \int_\eta \underbrace{d\lambda}_{=0} = \int_\sigma \lambda.$$

It is clearly a homomorphism. The image set of this homomorphism is called the *group of periods of* λ. We say λ has *rational periods* if the group of periods contains only rational numbers.

Lemma 1.18. *Any closed C^k-1-form, $0 \leq k \leq \infty$, can be C^0-approximated by a closed C^k-1-form with rational periods.*

Proof. Take closed smooth 1-forms $\mu_1, \dots, \mu_l$ representing a basis of $H^1(M,\mathbb{Z}) < H^1(M,\mathbb{R})$. By the Universal Coefficients Theorem, we can take smooth loops $\gamma_1, \dots, \gamma_l$ representing a basis of the free part of $H_1(M,\mathbb{Z})$ with $\int_{\gamma_i} \mu_j = \delta_{ij}$. Let λ be a closed C^k-1-form and fix a point $p_0 \in M$. Then

$$\lambda = \sum_{j=1}^{l} c_j \mu_j + df,$$

where $c_j = \int_{\gamma_j} \lambda$ and $f \colon M \to \mathbb{R}$ is the C^{k+1}-function given by

$$f(p) = \int_{p_0}^{p} (\lambda - \sum_{j=1}^{l} c_j \mu_j).$$

The integral defining f does not depend on the choice of smooth path from p_0 to p because $\lambda - \sum_{j=1}^{l} c_j \mu_j$ gives zero when integrated over a smooth loop. Indeed, by definition of the coefficients c_j, integrating over a smooth loop lying in the homology class of a combination of concatenations of $\gamma_1, \dots, \gamma_l$ gives zero, and integrating over a different loop gives zero because such a loop does not represent a homology class in the free part of $H_1(M,\mathbb{Z})$. Approximate the real numbers $c_1, \dots, c_l$ by rational numbers $c_1', \dots, c_l'$ and define

$$\lambda' = \sum_{j=1}^{l} c_j' \mu_j + df.$$

Then λ' is a C^k-form that is C^0-close to λ. Moreover, integrating over a combination of concatenations of $\gamma_1, \dots, \gamma_l$ gives a rational number obtained from addition of $c_1', \dots, c_l'$ (possibly with repetition) while integrating over a different loop gives zero, as before. Thus, λ' has rational periods. □

Next, we turn to the interplay of foliations and 1-forms. A foliation $\mathcal{F}$ is *determined by a 1-form* λ if $T\mathcal{F} = \ker(\lambda)$. If the form is closed, then this has a consequence on the holonomy of the foliation:

Lemma 1.19. *Suppose $\mathcal{F}$ is a foliation determined by a closed 1-form. Then $\mathcal{F}$ has trivial holonomy.*

Proof. Suppose for contradiction, this is not true. Then there is a loop γ contained in some leaf of $\mathcal{F}$ such that the holonomy induced by γ is non-trivial. Take a small transversal to the foliation through some point on γ. Then the holonomy of γ induces a non-trivial diffeomorphism of this transversal. In particular, we can find a point on the transversal close to γ which gets mapped to a different point on the transversal under the holonomy and by choosing the point sufficiently close we can ensure that the path of this point under the transport defining the holonomy is close to γ. Denote the path running from the original point to its image point by δ_1. Then δ_1 is contained in a single leaf by definition of the transport. Connect the endpoints of δ_1 by a path δ_2 transverse to the foliation. Denote the closed 1-form determining the foliation by λ. Note that integrating λ along δ_2 yields a non-zero number by transversality. Because δ_1 is close to γ but contained in a different leaf, the concatenation $\delta_1 * \delta_2$ can be taken so that γ and $\delta_1 * \delta_2$ bound an annulus. In particular, since λ is closed, Stoke's theorem yields

$$\int_\gamma \lambda = \int_{\delta_1 * \delta_2} \lambda = \int_{\delta_1} \lambda + \int_{\delta_2} \lambda.$$

This is a contradiction because the integrals over γ and δ_1 must both be zero since each of these paths is contained in a single leaf which integrates $\ker(\lambda)$. $\square$

We included this lemma about holonomy in order to prove the next result, which provides us with a useful tool to detect suspensions. How exactly this is related to suspensions will be discussed afterwards.

Proposition 1.20. *Suppose λ is a closed 1-form with rational periods. Then M is a fiber bundle over S^1 whose fibers are the leaves of the foliation determined by λ. If λ is C^k, $0 \leq k \leq \infty$, then the bundle map is C^{k+1}. Moreover, λ is a positive multiple of the pullback of the canonical volume form on S^1.*

Proof. Since $H_1(M, \mathbb{Z})$ surely is finitely generated by compactness of M, a positive multiple λ' of λ has only integer periods. Now fix a point $p_0 \in M$ and define $\Pi\colon M \to S^1$ by $\Pi(p) = \int_{p_0}^p \lambda' \pmod 1$. This integral is independent of the smooth path from p_0 to p because λ' takes integer values when integrated over smooth loops. Moreover, Π inherits the smoothness from λ' with one additional degree of smoothness and has differential $d\Pi(X) = \lambda'(X)$. In other words, λ' is the pullback of the canonical volume form on S^1. Π maps two different points p and q from the same leaf to the same point in S^1 because we can compute $\Pi(q)$ with a path from p_0 to q via p, where we go from p to q inside the leaf. Then the second part of the path has no contribution to the integral since the leaf integrates the kernel of λ'. Therefore, $\Pi^{-1}(z)$ is always a union of leaves for any $z \in S^1$. If there were infinitely many leaves in one level set of Π, then there would be an accumulation of leaves on which Π is constant. However, when we take a path transverse to the leaves, then the integral of λ' along this path increases. Thus, no such accumulation can exist. Hence, the level sets can contain only finitely many leaves, which must be compact because $\Pi^{-1}(z)$ is a closed subset of the compact manifold M. We almost have a bundle structure but each base-point admits several leaves above it. What we need to do is "unwind" the base. This can be done because the holonomy of the foliation vanishes by the last lemma, which is known as the Reeb Stability Theorem (more precisely, we use Thurston's generalization of that theorem, see [Thu74, Thrm. 1+2]). $\square$

We need to specify how the previous proposition is linked to suspensions, which boils down to arguing why the above bundle structure comes from a suspension. To do so, we introduce the notion of a section of a flow. A *section* of a flow is a codimension one submanifold that is transverse to the flow direction such that any orbit starting in this submanifold returns to it in finite time in both forward and backward direction. Sections are the main tool to detect suspensions. Indeed, if a flow admits a section, then there is a time-change so that the first return map of the section is given by the time-1 map of the scaled flow. Then this time-change is the suspension flow of said first return map of the section.

Corollary 1.21. *If a flow is transverse to the kernel of some non-trivial closed 1-form, then it is a time-change of a suspension flow.*

Proof. By lemma 1.18, we can replace the given 1-form by a closed 1-form λ with rational periods. Taking λ C^0-close to the original form, the flow remains transverse to the kernel of λ. Then proposition 1.20

provides us with a bundle structure over a circle whose fibers are the leaves of the foliation integrating $\ker(\lambda)$. We claim that any fiber is a section. Transversality to the flow direction is clear since the fiber is a leaf integrating $\ker(\lambda)$ and since the flow is transverse to $\ker(\lambda)$ by hypothesis. Now assume for contradiction that some orbit starting in the fiber does not return. Then the projection of this orbit to the base circle stays in some contractible subinterval. Therefore, it must admit a limit point. But this contradicts that the fiber above this limit point is transverse to the flow direction. Thus, any fiber is a section. A priori, these sections are only C^{k+1}, where C^k is the smoothness of the closed 1-form. However, we can always perturb such a section to a smooth one. □

In the previous results, we did not specify any particular 1-form to work with. However, in the Anosov case, there is a distinguished 1-form associated with the flow. Any Anosov flow has an *associated 1-form* defined by being 0 on $E^s \oplus E^u$ and being 1 on the infinitesimal generator. Note that this form is in general only (Hölder-)continuous.

Corollary 1.22. *For an Anosov flow, its associated 1-form is closed if and only if the bundle $E^s \oplus E^u$ is integrable. In this case, the flow is a time-change of a suspension of an Anosov diffeomorphism.*

Proof. By definition, the flow is transverse to the kernel of its associated 1-form. Denote the latter by λ. By theorem 1.17, $E^s \oplus E^u$ is integrable if and only if $d\lambda$ exists and $\lambda \wedge d\lambda$ is zero. Further, this implies $d\lambda = \iota_F(\lambda \wedge d\lambda) = 0$ because $0 = \mathcal{L}_F\lambda = \iota_F d\lambda$. In this case, we can apply corollary 1.21 to deduce that a time-change of the flow is a suspension. Such a time-change remains Anosov and we conclude with proposition 1.14. □

1.6 Contact Flows

Another class of examples in a way "opposite" to suspensions is formed by contact flows. A flow ϕ_t is *contact* if there exists a ϕ_t-invariant contact form λ with $\lambda(F) = 1$. Note that a 1-form λ with $\iota_F\lambda = 1$ is ϕ_t-invariant if and only if $\iota_F d\lambda = 0$ by Cartan's formula. If such a form λ exists and if the flow is Anosov, then necessarily $\ker(\lambda) = E^s \oplus E^u$. Thus, an Anosov flow is contact if and only if its associated 1-form is a contact form. Let us briefly recall an important result by Poincaré on measure-preserving flows. Afterwards, we will explore an application of this result to contact flows.

Theorem 1.23 (Poincaré Recurrence Theorem)**.** *If ϕ_t is measurable with respect to some σ-algebra $\mathcal{A}$ and preserves a finite measure μ on $\mathcal{A}$, then for any $A \in \mathcal{A}$ and any $T \geq 0$ there exists a measurable subset $R_{A,T}$ of A of measure $\mu(R_{A,T}) = \mu(A)$ such that for every $x \in R_{A,T}$ there is a time $t \geq T$ with $\phi_t(x) \in A$.*

Proof. Consider the set

$$R_{A,T} = \bigcup_{k \geq 1} \phi_{-kT}(A),$$

which is measurable by the hypothesis on ϕ_t, and define $B = A \setminus R_{A,T}$. Then $x \in \phi_{-jT}(B)$ if and only if $\phi_{jT}(x) \in A$ and if for all $k \geq 1$ we have $\phi_{(j+k)T}(x) \notin A$. In particular, $\phi_{-iT}(B)$ and $\phi_{-jT}(B)$ are disjoint for $i \neq j$. Since μ is finite and preserved by ϕ_t, it follows that $\mu(B) = 0$. □

Corollary 1.24. *A contact Anosov flow is transitive.*

Proof. If λ denotes the associated 1-form of the flow, then $\lambda \wedge (d\lambda)^n$ defines a ϕ_t-invariant finite measure on the Borel-σ-algebra on M via integration. By the Poincaré Recurrence Theorem, the non-wandering set is M. Thus, the Spectral Decomposition is trivially $\{M\}$ and the statement follows. □

Now that we established transitivity of contact Anosov flows, corollary 1.16 comes into play. Namely, a contact Anosov flow is either mixing or a suspension. As mentioned in the beginning, contact flows are supposed to be in a way "opposite" to suspensions. Let us verify this:

Lemma 1.25. *A contact Anosov flow is never a suspension. In dimension three, no time-change of a contact Anosov flow is a suspension.*

Proof. As used before, if it was a suspension, then the tangent bundle of the fibers from the suspension structure must coincide with $E^s \oplus E^u$. Then the associated 1-form must be exactly the pull-back of the canonical volume form on the base circle of the suspension structure. However, the former is contact while the latter is closed. This proves the first statement. Now suppose we are in dimension three and there is a contact Anosov flow with a time-change that is a suspension. Let λ and λ' denote the associated 1-forms, respectively, so that $\lambda \wedge d\lambda$ is a volume form and λ' is closed. Then $\lambda' \wedge d\lambda$ is a volume form, as well. This form is exact with primitive $\lambda \wedge \lambda'$, but, being a closed manifold, M cannot admit an exact volume form by Stoke's theorem. □

Corollary 1.26. *A contact Anosov flow is topologically mixing.*

A standard example of a contact flow is the geodesic flow on a Riemannian manifold. Indeed, if λ_0 denotes the canonical 1-form on the unit tangent bundle UM (see the appendix), then

$$(\lambda_0)_{(x,u)}(G) = \langle u, G_H \rangle = 1,$$
$$(d\lambda_0)_{(x,u)}(G, X) = -\langle u, X_V \rangle = 0$$

because $X_V \in \operatorname{span}\langle Ju \rangle$, where J denotes the almost complex structure associated to the Riemannian metric. The previous corollary together with theorem 1.6 yields:

Corollary 1.27. *Geodesic flows on Riemannian manifolds of strictly negative sectional curvature are topologically mixing.*

Since time-changes have repeatedly popped up recently, let us have a closer look at them and their role in the class of contact flows. The proof of proposition 1.1 shows how a time-change affects the Anosov splitting. Since smoothness of the associated 1-form is linked to smoothness of the bundle $\ker(\lambda) = E^s \oplus E^u$, we cannot expect a general time-change to preserve the contact property of a contact Anosov flow. For a certain class of time-changes, this can be rectified, though. We say a time-change is *canonical* if the infinitesimal generator of the new flow is given by $\frac{1}{c+\alpha(F)}F$ for some closed 1-form α and a constant c with $c + \alpha(F)$ never zero.

Proposition 1.28. *In dimension three, a time-change of a contact Anosov flow preserves the contact property if and only if the time-change is canonical.*

Proof. Denote by λ the associated 1-form of the contact Anosov flow we start with. If the time-change is canonical with $F' = \frac{F}{c+\alpha(F)}$ for a closed 1-form α and a constant c, then $c\lambda + \alpha$ is a contact form with Reeb vector field F'. This shows one direction. Now assume F' is any time-change so that the associated 1-form λ' of the new flow is a contact form. Let f denote the function satisfying $F = fF'$. Then $\iota_{F'}(f\lambda \wedge d\lambda) = d\lambda$ shows that $f\lambda \wedge d\lambda$ is invariant under the time-change ϕ'_t. Both being volume forms, $\lambda' \wedge d\lambda'$ must be a multiple of $f\lambda \wedge d\lambda$ by some smooth function c. This function is ϕ'_t-invariant since both volume forms are. Thus, c is constant along orbits and since transitivity implies the existence of a dense orbit, we can conclude that c is constant. Then

$$d\lambda' = \iota_{F'}(\lambda' \wedge d\lambda') = \iota_{F'}(cf\lambda \wedge d\lambda) = cd\lambda.$$

In particular, $\alpha = \lambda' - c\lambda$ is closed, and α and c are as needed. □

Contact flows are a special case of *volume-preserving flows*, i.e. flows that preserve a volume form. Even more general, we may consider flows preserving a top-dimensional form that may not even be a volume form. First of all, we can easily upgrade corollary 1.24 with the same proof:

Proposition 1.29. *If an Anosov flow preserves a (continuous) non-trivial top-dimensional form, then it is transitive.*

Proof. Let Ω denote the non-wandering set and ω such a top-dimensional ϕ_t-invariant form. Since ω is not identically zero, it defines a finite ϕ_t-invariant measure on the Borel-σ-algebra by integration. Denote by S the set of points where ω vanishes. By the Poincaré Recurrence Theorem, $M \setminus S \subset \Omega$. Therefore, Ω contains an open set. By lemma 1.11, this enforces the Spectral Decomposition to be trivial, which implies the statement. □

For the next result, recall that given a Riemannian metric g there is a unique volume form vol_g associated to this metric. The *Riemannian measure of* g is then given by $\mu_g(A) = \int_A \mathrm{vol}_g$.

Proposition 1.30. *Let ϕ_t be an Anosov flow and ω be a (continuous) ϕ_t-invariant top-dimensional form. Denote by S the set of points where ω vanishes. Then either $S = M$ or S has Riemannian measure zero for any Riemannian metric on M.*

In order to prove this, we will first investigate the set $R = \{p \in M \mid \lim_{t\to\infty} J_t(p) = \infty\}$, where $J_t(p) = \det(d\phi_t)_p$ is the Jacobian.

Lemma 1.31. *Given an Anosov flow and any Riemannian metric on M, the set R has Riemannian measure zero and is saturated by the global strong stable foliation.*

Proof. Suppose for contradiction $\mu_g(R) > \delta > 0$ for some Riemannian metric g. Then Egorov's theorem implies the existence of a compact set $K \subset R$ of measure $\mu_g(K) > \delta$ on which the functions J_n diverge uniformly (see theorem A.2). But the uniform divergence implies

$$\infty > \mu_g(M) \geq \mu_g(\phi_n(K)) = \int_{\phi_n(K)} \mathrm{vol}_g = \int_K \phi_n^* \mathrm{vol}_g \geq \left(\inf_{p\in K} J_n(p)\right) \mu_g(K) \to \infty.$$

This proves the first assertion. For the second statement, note that for any point $p \in M$, any time $t = N + s$, and with $p' = \phi_s(p)$ we have

$$\begin{aligned}
\log(J_t(p)) &= \log\left(\det\left((d\phi_1)_{\phi_{N-1}(p')} \circ \cdots \circ (d\phi_1)_{p'} \circ (d\phi_s)_p\right)\right) \\
&= \sum_{k=0}^{N-1} \log(\det((d\phi_1)_{\phi_k(p')})) + \log(\det((d\phi_s)_p)) \\
&= \sum_{k=0}^{N-1} \log(J_1(\phi_k(p'))) + \log(J_s(p)).
\end{aligned}$$

By compactness, $\log(J_t(p))$ as a function of $[0,1]\times M$ is Lipschitz continuous with some Lipschitz constant L. Now take any $p_0 \in R$. Given any point $q_0 \in W^s(p_0)$, we may take iterates $q = \phi_T(q_0)$ and $p = \phi_T(p_0)$ so that q lies in a local strong stable manifold $W^s_\epsilon(p)$ for some small $\epsilon > 0$. This implies the existence of constants $C \geq 1$ and $0 < \mu < 1$ with $d_g(\phi_t(p), \phi_t(q)) \leq C\mu^t\epsilon$. Therefore, we can estimate for $t = N + s$

with $0 \leq s < 1$, $p' = \phi_s(p)$, and $q' = \phi_s(q)$

$$\begin{aligned}|\log(J_t(p)) - \log(J_t(q))| &\leq \sum_{k=0}^{N-1} |\log(J_1(\phi_k(p'))) - \log(J_1(\phi_k(q')))| + |\log(J_s(p)) - \log(J_s(q))| \\ &\leq \sum_{k=0}^{N-1} L d_g(\phi_k(p'), \phi_k(q')) + L d_g(p,q) \\ &\leq \sum_{k=0}^{N-1} L C \mu^k \epsilon + L\epsilon \leq LC\frac{2-\mu}{1-\mu}\epsilon.\end{aligned}$$

By definition of $p_0 \in R$,

$$\log(J_t(p)) = \log(J_{t+T}(p_0)) - \log(J_T(p_0)) \xrightarrow{t\to\infty} \infty.$$

Therefore, we also have

$$\log(J_{t+T}(q_0)) = \log(J_t(q)) + \log(J_T(q_0)) \xrightarrow{t\to\infty} \infty$$

and, hence, $q_0 \in R$. This concludes $W^s(p_0) \subset R$, as desired. □

Proof of proposition 1.30. Suppose that $\mu_g(S) > 0$ for some Riemannian metric g on M. By the previous lemma, $\mu(M \setminus R) = 1$ and, hence, the set $S \cap (M \setminus R)$ is not empty. Take any point p in this set. Also by the previous lemma, $W^s(p) \subset M \setminus R$. We claim that $W^s(p)$ is contained in S, as well. Then $W^{sc}(p) \subset S$ by invariance of ω. Proposition 1.29 and lemma 1.12 imply that $W^{sc}(p)$ is dense in M. Given the claim and that S is closed, we conclude $S = M$. It remains to prove the claim. Since $p \in S$, $\omega_{\phi_t(p)} = 0$ for all times. Take any point $q \in W^s(p)$. By definition, $\phi_t(p)$ and $\phi_t(q)$ converge to each other, so $||\omega_{\phi_t(q)}|| \to 0$ as $t \to \infty$. $J_{t_n}(q)$ is bounded on some subsequence $(t_n)_{n\geq 0}$ because $q \in M \setminus R$. Hence, $J_{t_n}(q)||\omega_{\phi_{t_n}(q)}|| \to 0$ as $n \to \infty$ and it suffices to verify $||\omega_q|| = J_t(q)||\omega_{\phi_t(q)}||$ for all $t \in \mathbb{R}$. The set of times t for which the equality holds is clearly closed and we will show that it is also open. Take local (continuous) vector fields X^c, X^s, X^u that span E^c, E^s, E^u, respectively, and are normalized by the metric to have norm 1. By invariance, there are (continuous) functions c, s, u with $d\phi_t(X^c) = c(t)X^c$ and similarly for X^s and X^u. Then

$$\omega_q(X^c, X^s, X^u) = c(t)s(t)u(t)\omega_{\phi_t(q)}(X^c, X^s, X^u)$$

and the problem reduces to showing $J_t(q) = c(t)s(t)u(t)$. If the vector fields were induced by coordinates on M, then $d\phi_t$ would take the form of a diagonal matrix with entries $c(t), s(t), u(t)$ in these coordinates and we would be done. Even though, in general, such coordinates do not exist due to regularity issues, we can smoothly approximate such coordinates to obtain the same conclusion. □

The payoff of studying these flows that preserve a top-dimensional form is a very useful way in dimension three for checking whether an Anosov flow is contact. Namely, we only need check smoothness of the associated 1-form and rule out the suspension case; we may skip verifying whether the associated 1-form really is a contact form:

Theorem 1.32. *Suppose we are in dimension three and have an Anosov flow whose associated 1-form λ is smooth. If λ is not closed, then the flow is contact.*

Proof. We need to show that $\lambda \wedge d\lambda$ defines a volume form. Let vol be any volume form on M. Then there exists a smooth function f with $\lambda \wedge d\lambda = f\mathrm{vol}$. This function is not constantly zero for otherwise

$d\lambda = \iota_F(\lambda\wedge d\lambda) = 0$, where we used $\iota_F d\lambda = \mathcal{L}_F\lambda = 0$. By proposition 1.30, the set $f^{-1}(0)$ has Riemannian measure zero for any Riemannian metric. Define a measurable function by

$$L(p) = \begin{cases} -\log(f(p)), & \text{if } f(p) > 0, \\ 0, & \text{if } f(p) = 0, \\ \log(f(p)), & \text{if } f(x) < 0. \end{cases}$$

Since $\lambda \wedge d\lambda$ is invariant under the flow, the function f is either strictly positive, constantly zero, or strictly negative along a fixed orbit. Thus, restricted to a fixed orbit, the function L is given by either $\log(f(p))$, $-\log(f(p))$, or 0. In particular, L is differentiable along the flow direction. Next, we use invariance of $\lambda \wedge d\lambda$ to compute

$$0 = \mathcal{L}_F(f\text{vol}) = f(\mathcal{L}_F\text{vol}) + df(F)\text{vol} = (fD + df(F))\text{vol},$$

where D is the divergence of F with respect to vol, meaning that D is the smooth function specified by $\mathcal{L}_F\text{vol} = D\text{vol}$. Then $dL(F) = -\frac{1}{f}df(F) = D$ on the set $f^{-1}(\mathbb{R}\setminus\{0\})$ of full Riemannian measure. By a theorem of Livsic ([HK90, Thrm. 2.1]), the function L has to be continuous. This implies that there can be no point with $f(p) = 0$. □

Remark 1.33. *One can also show that it suffices to assume that λ is C^1. Being in dimension three, this enforces λ to be C^∞. For a proof of this fact, see [HK90, Thrm. 2.3].*

1.7 Hamiltonian Structures

In this chapter, we generalize the setup provided by contact flows. Suppose M has dimension $2n+1$ and comes equipped with a 2-form Ω. We call (M,Ω) a *Hamiltonian structure* if Ω is closed and the kernel of Ω is an orientable 1-dimensional distribution. We call a vector field spanning this distribution a *Reeb vector field* of the Hamiltonian structure and its flow a *Reeb flow*. Note that any Hamiltonian structure admits many Reeb flows but any two Reeb flows are a time-change of each other. There are two interesting specializations. A Hamiltonian structure (M,Ω) is *stable* if there exists a 1-form λ with $\ker(\Omega) \subset \ker(d\lambda)$ and such that $\lambda \wedge \Omega^n$ is a volume form. Such a 1-form is called a *stabilizing 1-form* for (M,Ω). When a stabilizing 1-form λ is fixed, then there is a distinguished Reeb vector field F specified by $\lambda(F) = 1$. We speak of *the* Reeb vector field and *the* Reeb flow if the stabilizing 1-form is understood. Further, (M,Ω) is *HS-contact* if it admits a primitive λ of Ω so that $\lambda \wedge \Omega^n$ is a volume form. In particular, such a Hamiltonian structure is always stable. As a motivation for introducing stable Hamiltonian structures as a generalization of contact structures, let us remark that the Weinstein conjecture holds true for any stable Hamiltonian structure in dimension three assuming that M is not a torus bundle over the circle ([HT09, Thrm. 1.1]).

Remark 1.34. *We use the word "HS-contact" due to the following interplay of notation, which might be misleading: Suppose we fix a Reeb flow ϕ_t of a Hamiltonian structure (M,Ω). Then*

$$(M,\Omega) \textit{ is HS-contact } \Rightarrow \phi_t \textit{ admits a time-change that is contact } \Rightarrow (M,\Omega) \textit{ is stable.}$$

In the first implication, we cannot guarantee that ϕ_t itself is contact. Furthermore, in general, each implication cannot be reversed. For instance, we cannot conclude from ϕ_t being contact that (M,Ω) is HS-contact (due to the restriction $d\lambda = \Omega$).

Since any two Reeb flows of a Hamiltonian structure (M,Ω) are time-changes of each other, we can call (M,Ω) *Anosov* if some (any) Reeb flow is Anosov. Our main interest focuses on Anosov stable Hamiltonian structures. An obvious question is when a stable Hamiltonian structure is HS-contact. In the Anosov case, there are some conditions on the Anosov splitting under which stability implies

HS-contact, see [MP10, Thrm. A]. In dimension three, the picture simplifies greatly. This is due to $d\lambda$ necessarily being a multiple of Ω by some smooth function. Let us exploit the 3-dimensional case in more detail. Suppose M is a surface bundle over S^1. Then any non-degenerate 2-form on the surface gives rise to a Hamiltonian structure on M and the pull-back of any volume form on S^1 is a stabilizing 1-form. This stabilizing 1-form is closed. We know from the section on suspension flows that the converse holds, too:

Proposition 1.35. *Suppose (M, Ω) is a stable Hamiltonian structure in dimension three that admits a closed stabilizing 1-form. Then M admits the structure of a fiber bundle over S^1 with the following property: The pull-back λ of the canonical volume form on S^1 to M is a stabilizing 1-form for Ω. Furthermore, if ϕ_t denotes the Reeb flow of (λ, Ω), then (M, ϕ_t) is a suspension.*

Proof. The proof is essentially the one of corollary 1.21 but that we have a stable Hamiltonian structure gives us more information. Before, we could only qualitatively say that there is a time-change that is a suspension. With the stable Hamiltonian structure, we can pinpoint this time-change. We reproduce the argument here. Take any closed stabilizing 1-form for (M, Ω). Lemma 1.18 states that we can approximate this form (in the C^0-topology) by a closed 1-form λ' with rational periods. If we are sufficiently close, then the new form λ' remains a stabilizing 1-form for (M, Ω). Proposition 1.20 provides the bundle structure over S^1 whose fibers are the leaves of the foliation integrating $\ker(\lambda')$ (which exists by theorem 1.17). Moreover, it states that λ' is a positive multiple of λ (which, in particular, implies that λ is a stabilizing 1-form for (M, Ω)). Let L be a fiber, i.e. a leaf of the foliation integrating $\ker(\lambda') = \ker(\lambda)$. We claim that L is a section of ϕ_t. Indeed, the transversality is clear because L integrates the kernel of λ and ϕ_t is the Reeb flow of the latter. Secondly, since λ is the pullback of the canonical volume form on S^1, the projection of the flow to the base must be the translation $x \mapsto x + t \pmod 1$. Therefore, any orbit starting in L returns to L after time 1. Given the claim, ϕ_t must be the suspension flow of the diffeomorphism of L given by the time-1 map. □

Being able to detect suspensions, we can give the complete (simple) classification for the Anosov case in dimension three:

Theorem 1.36. *Any Anosov stable Hamiltonian structure in dimension three is either a suspension or HS-contact.*

Proof. Suppose (M, Ω) is an Anosov stable Hamiltonian structure. If it admits a closed stabilizing 1-form, then we are in the suspension case by proposition 1.35. If not, let λ denote a stabilizing 1-form and ϕ_t the Reeb flow of (λ, Ω). By definition, there is a function f with $d\lambda = f\Omega$. Since both λ and Ω are ϕ_t-invariant, so is f. Proposition 1.29 tells us that ϕ_t is transitive because $\lambda \wedge \Omega$ is a volume form. By transitivity and invariance, f must be constant. Thus, $\Omega = d(\frac{1}{f}\lambda)$ arose from the contact form $\frac{1}{f}\lambda$. □

Corollary 1.37. *When not in the suspension case, an Anosov Hamiltonian structure in dimension three is stable if and only if it is HS-contact.*

Remark 1.38. *In particular, in the Anosov case in dimension three, the terminology becomes a little less unfortunate because remark 1.34 evolves into:*

$$(M, \Omega) \text{ is HS-contact} \iff \phi_t \text{ admits a time-change that is contact.}$$

Moreover, any contact Reeb flow of (M, Ω) is a constant time-change of a Reeb flow induced by a stabilizing primitive of Ω.

Proof. Suppose ϕ_t is a contact Reeb flow of (M, Ω) with associated 1-form λ. Then $\lambda \wedge d\lambda = c\lambda \wedge \Omega$ for some smooth function c because both are volume forms. Further, both forms are ϕ_t-invariant and, hence, so is c. By transitivity of the flow (corollary 1.24), c must be constant. Contracting with F yields $d\lambda = c\Omega$ because $\iota_F d\lambda = \mathcal{L}_F \lambda = 0$. Thus, $\frac{1}{c}\lambda$ is a stabilizing primitive of (M, Ω). □

Corollary 1.39. *When not in the suspension case, an Anosov Hamiltonian structure* (M, Ω) *in dimension three cannot be stable if* Ω *is not exact.*

Remark 1.40. *The converse is not true. We will encounter an example of an exact Anosov Hamiltonian structure in dimension three that is not stable in remark 2.41.*

Sometimes, being HS-contact is too strong of a property to ask for. A weaker but useful notion is the following: A Hamiltonian structure (M, Ω) is *virtually contact* if there is a cover $\hat{M} \to M$ so that the lift $\hat{\Omega}$ admits a primitive λ with $||\lambda||_\infty < \infty$ and $\inf_{x \in \hat{M}} |\lambda_x(\hat{F}(x))| > 0$ in some (any) lifted metric, where $\hat{F}$ is the lift of some (any) Reeb vector field of (M, Ω). That $\lambda \wedge \hat{\Omega}^n$ is a volume form is implicit in the definition because $\lambda(\hat{F})$ never vanishes. In particular, if (M, Ω) is HS-contact, then it is virtually contact, and if it is virtually contact, then the lifted Hamiltonian structure $(\hat{M}, \hat{\Omega})$ is HS-contact. In general, the reverse implications do not hold as we will find counter-examples in corollary 2.44. In the next chapter, we will discuss in detail a class of examples of Hamiltonian structures and their respective specializations.

2 Magnetic Flows

2.1 Perturbing Geodesic Flows

In this chapter, we will perturb geodesic flows by introducing a magnetic field. This can be done on any manifold, but we are interested in the case of an oriented closed surface Σ equipped with a Riemannian metric g. We will use the notation $\langle \cdot, \cdot \rangle$ for the metric, denote its Gaussian curvature by K, and denote the corresponding Levi-Civita connection by ∇. Let λ_0 and ω_0 denote the canonical forms on $T\Sigma$ from the appendix and let σ denote a (necessarily closed) 2-form on Σ. The new 2-form

$$\omega = \omega_0 + \pi^* \sigma$$

is called a *twisted symplectic form*. Indeed, this is a symplectic form on $T\Sigma$ as it is obviously closed and we can easily check that it is also non-degenerate. If $\omega(X, Y)$ vanishes for all vectors Y on $T\Sigma$, then the horizontal component of X must be 0 since the right hand side in

$$\langle X_H, Y_V \rangle - \langle X_V, Y_H \rangle = -\sigma (X_H, Y_H)$$

is independent of Y_V. But then $\omega(X, Y) = 0$ for all Y amounts to $\omega_0(X, Y) = 0$ for all Y, which implies $X = 0$ by non-degeneracy of ω_0. Thus, a twisted symplectic form is symplectic. A smooth function $T^*\Sigma \cong T\Sigma \to \mathbb{R}$ is called a *Hamiltonian*. Usually, a suitable class of Hamiltonians is given by convex and super-linear ones, but we will restrict our attention to the *energy Hamiltonian* given by

$$E \colon T\Sigma \to \mathbb{R}, \; E(x, v) = \frac{1}{2} \langle v, v \rangle$$

since it captures all the key concepts. By non-degeneracy, there exists a unique vector field F, namely the symplectic gradient of E with respect to ω, specified by the property $\iota_F \omega = dE$. The flow ϕ_t generated by F is called a *magnetic flow* (or *twisted geodesic flow*). Note that ϕ_t preserves both the Hamiltonian and the twisted symplectic form as can be seen from Cartan's formula:

$$\mathcal{L}_F E = \iota_F dE = \iota_F^2 E = 0 \quad \text{and} \quad \mathcal{L}_F \omega = d\iota_F \omega = d^2 E = 0.$$

We can use non-degeneracy once more to find a unique bundle map $Y \colon T\Sigma \to T\Sigma$, called the *Lorentz force*[1], that satisfies

$$\sigma_x(u, v) = \langle Y_x(u), v \rangle.$$

We then get a representation of dE by

$$\begin{aligned} dE(X) = \omega(F, X) &= \langle F_H, X_V - Y(X_H) \rangle - \langle F_V, X_H \rangle, \\ &= \langle F_H, X_V \rangle - \langle F_V - Y(F_H), X_H \rangle. \end{aligned}$$

Since we know that $(dE)_{(x,v)}(X)$ is just $\langle v, X_V \rangle$, we conclude that the vector field F can be written in terms of the Lorentz force as

$$F(x, v) = (v, Y_x(v)) \, .$$

We have not yet used that Σ is two-dimensional and oriented. Since Y must map a vector v into its orthogonal complement $v^\perp$, we can write the Lorentz force as

$$Y \colon T\Sigma \to T\Sigma, \; Y_x(v) = s(x) J v,$$

[1] The letter 'L' will be used later for the Lagrangian.

where J is the almost complex structure of g that rotates a vector counter-clockwise by angle $\pi/2$ and s is some smooth function on Σ. Therefore, if Ω_{area} denotes the area form of (Σ, g), i.e. the form $(\Omega_{\text{area}})_x(u,v) = \langle Ju, v\rangle$, then we get

$$\sigma = s\Omega_{\text{area}}.$$

We call s the *magnetic magnitude.* Note that $s \equiv 0$ corresponds to the geodesic flow. Since the Hamiltonian is constant along orbits of ϕ_t, the flow induces a well-defined restriction to any energy level of E. Denote the level of energy k by $S_k = E^{-1}(k)$. Let $\iota\colon S_k \hookrightarrow T\Sigma$ denote the inclusion and set $\Omega = \iota^*\omega$. Note that (S_k, Ω) defines a Hamiltonian structure and the magnetic flow is one of its Reeb flows. We will abusively write λ_0 and ω_0 for $\iota^*\lambda_0$ and $\iota^*\omega_0$. Furthermore, from now on π denotes the projection $S_k \to \Sigma$, i.e. we replace the previous π by $\pi \circ \iota$. Lastly, for convenience, functions like s and K that are defined on Σ may be regarded as functions on S_k by viewing them as $s \circ \pi$ and $K \circ \pi$. Next, we would like to analyze the cohomology class of Ω. Introduce new 1-forms on S_k by

$$\mu_{(x,v)}(X) = \langle Jv, X_H\rangle \quad \text{and} \quad \psi_{(x,v)}(X) = \langle Jv, X_V\rangle.$$

Proposition 2.1 (Cartan's structural equations)**.** *On the energy level S_k, the forms λ_0, μ, and ψ satisfy*

$$\begin{aligned} 2kd\lambda_0 &= \psi \wedge \mu, \\ 2kd\mu &= -\psi \wedge \lambda_0, \\ d\psi &= -K\lambda_0 \wedge \mu. \end{aligned}$$

Proof. We already know that $d\lambda_0 = -\omega_0$. Moreover, an analogue computation as in the appendix shows that

$$d\mu(X,Y) = \langle JX_H, Y_V\rangle + \langle JX_V, Y_H\rangle.$$

Using that X_V and Y_V live in $\operatorname{span}\langle Jv\rangle$ for $X, Y \in T_{(x,v)}S_k$, we find $r, s \in \mathbb{R}$ with $X_V = rJv$ and $Y_V = sJv$. If $r, s \neq 0$, then

$$\begin{aligned} (\psi \wedge \mu)_{(x,v)}(X,Y) &= \langle Jv, X_V\rangle\langle Jv, Y_H\rangle - \langle Jv, Y_V\rangle\langle Jv, X_H\rangle \\ &= \langle Jv, rJv\rangle\langle \frac{1}{r}X_V, Y_H\rangle - \langle Jv, sJv\rangle\langle -\frac{1}{s}Y_V, X_H\rangle \\ &= -|v|^2\omega_0(X,Y), \end{aligned}$$

$$\begin{aligned} -(\psi \wedge \lambda_0)_{(x,v)}(X,Y) &= -\langle Jv, X_V\rangle\langle v, Y_H\rangle + \langle Jv, Y_V\rangle\langle v, X_H\rangle \\ &= -\langle Jv, rJv\rangle\langle -\frac{1}{r}JX_V, Y_H\rangle + \langle Jv, sJv\rangle\langle -\frac{1}{s}JY_V, X_H\rangle \\ &= |v|^2 d\mu(X,Y). \end{aligned}$$

The computation is even simpler if r or s is zero. Since $|v|^2 = 2k$ on S_k, this establishes the first two equations. For the third, begin by observing $\langle v, w\rangle Jv - \langle Jv, w\rangle v = 2kJw$ for any $v \in S_k$. Hence, we find

$$\begin{aligned} (\lambda_0 \wedge \mu)_{(x,v)}(X,Y) &= \langle v, X_H\rangle\langle Jv, Y_H\rangle - \langle v, Y_H\rangle\langle Jv, X_H\rangle \\ &= \Big\langle \Big(\langle v, X_H\rangle Jv - \langle Jv, X_H\rangle v\Big), Y_H\Big\rangle \\ &= 2k\langle JX_H, Y_H\rangle = 2k\pi^*\Omega_{\text{area}}(X,Y). \end{aligned}$$

Thus, it suffices to show that $d\psi = -2kK\pi^*\Omega_{\text{area}}$. That this equation holds on the unit tangent bundle, i.e. for $k = \frac{1}{2}$, is due to ψ being the connection 1-form (by definition). Let $\iota_0\colon U\Sigma \hookrightarrow T\Sigma$ and $\iota\colon S_k \hookrightarrow T\Sigma$

denote the inclusions and consider the maps $h\colon S_k \to U\Sigma$ and $j\colon T\Sigma \to T\Sigma$ given by $h(x,v) = (x, \frac{v}{\sqrt{2k}})$ and the same formula for j so that $\iota = j^{-1} \circ \iota_0 \circ h$. Then

$$-K\iota^*\pi^*\Omega_{\mathrm{area}} = -Kh^*\iota_0^*\pi^*\Omega_{\mathrm{area}} = h^*\iota_0^* d\psi = \iota^* j^* d\psi.$$

Therefore, we only need to verify $j^*d\psi = \frac{1}{2k}d\psi$. Introduce vector fields on $T\Sigma$ by

$$G(x,v) = (v,0), \quad H(x,v) = (Jv,0), \quad V(x,v) = (0,Jv).$$

Note that they form a basis of TS_k. Restricted to S_k, they are the vector fields dual to λ_0, μ, and ψ, up to a factor $2k$. In particular, since we already know $d\psi = -K\lambda_0 \wedge \mu$ on $S_{\frac{1}{2}}$, we find $[G,H] = KV$ on $S_{\frac{1}{2}}$. Using

$$G \circ h^{-1} = \sqrt{2k}(dh^{-1})(G), \quad H \circ h^{-1} = \sqrt{2k}(dh^{-1})(H), \quad V \circ h^{-1} = (dh^{-1})(V),$$

we conclude that

$$[G,H] \circ h^{-1} = 2k(dh^{-1})([G,H]) = 2k(dh^{-1})(KV) = 2kK(V \circ h^{-1}),$$

i.e. $[G,H] = 2kKV$ on S_k. We are finished since

$$\begin{aligned}(d\psi)_{(x,v)}(G,H) &= -\psi_{(x,v)}([G,H]) = -\psi_{(x,v)}(|v|^2KV) = -|v|^4K,\\ (j^*d\psi)_{(x,v)}(G,H) &= -\psi_{j(x,v)}(|v|^2K(dj)(V)) = -|v|^2K\psi_{j(x,v)}(V \circ j) = -|v|^4\frac{K}{2k},\end{aligned}$$

and both terms are zero for any other input consisting of combinations of basis elements G, H, V. □

We partially already discussed the dual version of Cartan's structural equations for vector fields. G, H, and V are called the *geodesic*, the *horizontal*, and the *vertical vector field*, respectively. We noted that they are the vector fields dual to λ_0, μ, and ψ, up to a factor $2k$, when restricted to S_k.

Corollary 2.2 (Cartan's structural equations, dual version)**.** *On the energy level S_k, the vector fields G, H, and V satisfy*

$$\begin{aligned}[H,V] &= G,\\ [V,G] &= H,\\ [G,H] &= 2kKV.\end{aligned}$$

The interesting bit is to see how the magnetic flow depends on both the energy level k and the magnetic magnitude s. Actually, it suffices to only consider a single energy level. Indeed, suppose ϕ_t is the magnetic flow of $\omega_0 + s\pi^*\Omega_{\mathrm{area}}$ on the energy level S_k. Then ϕ_t is the flow generated by the vector field $(v, s(x)Jv)$. Denote by ϕ'_t the flow on $U\Sigma = S_{\frac{1}{2}}$ that is generated by $(u, s'(x)Ju)$, i.e. the magnetic flow of $\omega_0 + s'\pi^*\Omega_{\mathrm{area}}$. We will conjugate ϕ'_t by the map $h\colon U\Sigma \to S_k$ that sends (x,u) to $(x, \sqrt{2k}u)$. Given a point $(x,u) \in U\Sigma$, denote by $\gamma_{(x,u)}(t)$ the unique curve on Σ that satisfies

$$\gamma_{(x,u)}(0) = x, \quad \dot\gamma_{(x,u)}(0) = u, \quad \nabla_t\dot\gamma_{(x,u)}(0) = s'(x)Ju.$$

The existence of such curves is established as for geodesics. Then we can write the flow on $U\Sigma$ as

$$\phi'_t(x,u) = \left(\gamma_{(x,u)}(t), \dot\gamma_{(x,u)}(t)\right).$$

Further, the conjugated flow is

$$h \circ \phi'_t \circ h^{-1}(x,v) = \left(\gamma_{(x,\frac{v}{|v|})}(t), \sqrt{2k}\dot\gamma_{(x,\frac{v}{|v|})}(t)\right).$$

The infinitesimal generator of this conjugated flow is exactly

$$\frac{\partial}{\partial t}\Big|_{t=0}\left(h\circ\phi_t'\circ h^{-1}(x,v)\right)=\left(\frac{v}{|v|},\sqrt{2k}s'(x)J\frac{v}{|v|}\right)=\frac{1}{\sqrt{2k}}\left(v,\sqrt{2k}s'(x)Jv\right).$$

Thus, if we take $s'(x)=\frac{s(x)}{\sqrt{2k}}$, then this flow is a constant time change of ϕ_t. In the next subchapter, we will make use of this and carry out all the computations in the unit tangent bundle. However, afterwards we will switch back to the general point of view and vary the energy level while the magnetic form is fixed. Of course, the above also shows that this corresponds to scaling the magnetic magnitude by a constant factor.

Note that the proof of the third structural equation revealed $d\psi=-2kK\pi^*\Omega_{\text{area}}$. *For the next three subchapters, we add the standing assumption that Σ has genus at least two.* Then the Euler class $e=[K\Omega_{\text{area}}]$ is non-zero and, hence, it generates $H^2(\Sigma)$. In particular, σ is of the form $\kappa K\Omega_{\text{area}}+d\eta$ for some constant $\kappa\in\mathbb{R}$ and some 1-form η on Σ. Therefore, the last formula shows that, regardless of whether σ is exact, the restriction of the twisted symplectic form always is exact with

$$\Omega=-d\Big(\lambda_0+\frac{\kappa}{2k}\psi-\pi^*\eta\Big).$$

2.2 Example: Constant Curvature and Constant Magnitude

In this chapter, we will restrict our attention to the energy level $S_{\frac{1}{2}}=U\Sigma$ and we want to explore the special case in which both the magnetic magnitude s and the curvature K are constant. It is natural to ask whether the above primitive $\lambda=\lambda_0+\kappa\psi-\pi^*\eta$ of $-\Omega$ is a contact form. With $\kappa=s/K$, we now have $\sigma=\kappa K\Omega_{\text{area}}$, so we may take η to be zero. Then λ is reduced to $\lambda_0+\frac{s}{K}\psi$.

Proposition 2.3. *Suppose the magnetic magnitude and the curvature are constant. If $K+s^2\neq 0$, then $(U\Sigma,\Omega)$ is HS-contact with primitive $-\lambda=-(\lambda_0+\frac{s}{K}\psi)$. Further, the magnetic flow is contact with contact form $\frac{K}{K+s^2}\lambda$.*

Proof. If $K+s^2\neq 0$, then $\iota_F\lambda=1+\frac{s}{K}\langle Ju,F_V\rangle=\frac{K+s^2}{K}$ is never zero, so $\lambda\wedge\Omega$ is a volume form. □

Knowing that for most values of s and K the magnetic flow is contact, we can turn to the question when the magnetic flow is Anosov. We already know the answer for the geodesic flow, i.e. the case $s=0$. The subbundle spanned by H and V is exactly the kernel of λ_0. The kernel of $\lambda=\lambda_0+\frac{s}{K}\psi$ is spanned by H and $V_s=sG-KV$. The vector fields F,H, and V_s form a basis of $TU\Sigma$ if and only if $K+s^2\neq 0$. Thus, assume exactly $K+s^2\neq 0$ as we did in proposition 2.3. From the dual version of Cartan's structural equations we can deduce corresponding equations

$$\begin{aligned}
[H,V_s]&=-KG-sKV=-KF,\\
[V_s,F]&=-(K+s^2)H\ \ =-(K+s^2)H,\\
[F,H]&=-sG+KV\quad=-V_s.
\end{aligned}$$

Given some fixed $(x,u)\in U\Sigma$, abbreviate $F(t)=F\circ\phi_t(x,u)$ and likewise for H and V_s. Fix some initial vector $X\in TU\Sigma$. Then there are some smooth functions a, y, and z such that the differential of the magnetic flow can be written as

$$d\phi_t(X)=a(t)F(t)+y(t)H(t)+z(t)V_s(t).$$

Applying $d\phi_{-t}$ on both sides and differentiating with respect to time afterwards yields

$$\begin{aligned}
0=\;&\dot a(t)d\phi_{-t}(F(t))+a(t)d\phi_{-t}([F,F](t))\\
&+\dot y(t)d\phi_{-t}(H(t))+y(t)d\phi_{-t}([F,H](t))\\
&+\dot z(t)d\phi_{-t}(V_s(t))+z(t)d\phi_{-t}([F,V_s](t)).
\end{aligned}$$

Separating this into three equations, one for each basis element $d\phi_{-t}(F(t))$, $d\phi_{-t}(H(t))$, and $d\phi_{-t}(V_s(t))$, gives rise to the system of ordinary differential equations

$$\begin{aligned}0 &= \dot{a}(t),\\ 0 &= \dot{y}(t) + (K+s^2)z(t),\\ 0 &= \dot{z}(t) - y(t).\end{aligned}$$

Note that these equations hold without any assumption on Σ, ϕ_t, or Ω. Though, if we had not assumed that s and K are constant, then in the middle term it would actually read $s \circ \phi_t$ and $K \circ \phi_t$. Since we do not stand a chance of solving such equations, it is reasonable for this discussion that s and K are taken to be constant. If we assume in addition that $K + s^2$ is a negative number, then for the initial conditions $X_\pm = \pm\sqrt{-(K+s^2)}H \mp V_s$ we can find explicit solutions, namely

$$\begin{aligned}a_\pm(t) &= 0,\\ y_\pm(t) &= \pm\sqrt{-(K+s^2)}e^{\mp\sqrt{-(K+s^2)}t},\\ z_\pm(t) &= \mp e^{\mp\sqrt{-(K+s^2)}t}.\end{aligned}$$

In particular, the subbundles spanned by X_+ and X_- are invariant under $d\phi_t$. As the flow also possesses exponential growth on each subbundle, we find that ϕ_t is Anosov. In summary, we have proved the following result:

Proposition 2.4. *Suppose the magnetic magnitude and the curvature are constant. If $K + s^2 < 0$, then $(U\Sigma, \Omega)$ is Anosov. Moreover, the Anosov splitting of the magnetic flow is*

$$\begin{aligned}E^s &= \operatorname{span}\left\langle \sqrt{-(K+s^2)}H - V_s \right\rangle,\\ E^u &= \operatorname{span}\left\langle \sqrt{-(K+s^2)}H + V_s \right\rangle.\end{aligned}$$

Let us now turn to the case $K + s^2 > 0$. We wish to show that the magnetic flow is not Anosov in this case. We will argue by contradiction. Hence, assume there was some Anosov splitting $TU\Sigma = E^c \oplus E^s \oplus E^u$. Since the flow is contact by proposition 2.3, E^s and E^u must be contained in the kernel of λ, which is $\operatorname{span}\langle H, V_s\rangle$. We claim that neither E^s nor E^u may be equal to $\operatorname{span}\langle V_s\rangle$ at some point. If this was the case for, say, E^s at some point $p = (x,u)$, then the weak stable subspace E^{sc}_p equals $E^c_p \oplus \operatorname{span}\langle V_s(p)\rangle$. In particular, we had

$$V(p) = \frac{1}{K+s^2}\left(sF(p) - V_s(p)\right) \in E^{sc}_p,$$

but this contradicts the following transversality result:

Theorem 2.5. *If the magnetic flow is Anosov, then at no point $(x,u) \in U\Sigma$ is the vertical vector $V(x,u) = (0, Ju)$ contained in either the weak stable or the weak unstable subspace.*

Loosely speaking, this transversality result describes that we can not hope to achieve hyperbolic behavior merely by going around a fiber without moving the base-point. We postpone the proof of this theorem to the next chapter in order to stay focused on the current discussion. By the preceding argument, we can find smooth functions r_s and r_u on $U\Sigma$ such that $E^s = \langle H + r_s V_s\rangle$ and $E^u = \langle H + r_u V_s\rangle$. Take some initial vector $X = H(p) + r_s(p)V_s(p) \in E^s_p$. As before, we may take functions y and z with

$$d\phi_t(X) = y(t)H(t) + z(t)V_s(t).$$

Since $d\phi_t(X) \in E^s_{\phi_t(p)}$ by invariance, there also is a function R with

$$d\phi_t(X) = R(t)\left(H(t) + r_s(t)V_s(t)\right),$$

where $r_s(t) = r_s \circ \phi_t(p)$. Certainly, we must have $y(t) = R(t)$. Notice that $R(0) = 1$ and $R(t)$ can never be zero. Now recall the differential equations

$$\begin{aligned} 0 &= \dot{y}(t) + (K + s^2)z(t), \\ 0 &= \dot{z}(t) - y(t). \end{aligned}$$

Differentiating both equations and uncoupling them yields

$$0 = \ddot{R}(t) + (K + s^2)R(t).$$

However, we now concluded several properties of R that do not fit together: we know $R(t) > 0$, $\ddot{R}(t) < 0$, and $R(t)$ needs to decay exponentially as t grows since we started with $X \in E^s_p$. Such a function R does not exist. Let us summarize our findings.

Theorem 2.6. *Suppose the magnetic magnitude and the curvature are constant. If $K + s^2 \neq 0$, then $(U\Sigma, \Omega)$ is HS-contact with primitive $-\lambda = -(\lambda_0 + \frac{s}{K}\psi)$. Further, the magnetic flow is contact with contact form $\frac{K}{K+s^2}\lambda$. If, in addition, $K + s^2 > 0$, then $(U\Sigma, \Omega)$ is not Anosov. On the other hand, if $K + s^2 < 0$, then $(U\Sigma, \Omega)$ is Anosov and the Anosov splitting of the magnetic flow is*

$$\begin{aligned} E^s &= \operatorname{span}\left\langle \sqrt{-(K + s^2)}H - V_s \right\rangle, \\ E^u &= \operatorname{span}\left\langle \sqrt{-(K + s^2)}H + V_s \right\rangle. \end{aligned}$$

Remark 2.7. *If we are given a contact Anosov magnetic flow, then K and s must be constant (unless $s \equiv 0$). Thus, a posteriori, these two assumptions are both necessary and sufficient for the Anosov property in the contact case. We will prove this later in theorem 2.18.*

Remark 2.8. *We always worked with a higher genus surface. However, everything goes through unchanged for a sphere (we only excluded this case for convenience). In this case, $K + s^2 > 0$ is trivially satisfied and we obtain: No magnetic flow on a sphere (in particular, not the geodesic flow) is Anosov. Therefore, since we are mainly interested in Anosov flows in this thesis, we are not giving up on any interesting examples when we exclude the sphere case. Hence, we will continue to do so.*

We notice that the number $\sqrt{-K}$ plays a crucial role in determining the behavior of the magnetic flow. For a magnetic magnitude s below this critical value $\sqrt{-K}$ we get an Anosov system, while for larger magnitudes being Anosov is strictly excluded. This is a special instance of the more general notion of *Mañé's critical value*, which we will soon introduce and discuss in detail.
For completeness, let us quickly translate the above result into the setting for arbitrary energy levels via the conjugacy we mentioned in the first subchapter. Introduce yet a new vector field on $T\Sigma$ by $V_{s,k}(x, v) = (sv, -2kKJv)$. Then theorem 2.6 for energy levels reads:

Theorem 2.9. *Suppose the magnetic magnitude and the curvature are constant. If $2kK + s^2 \neq 0$, then (S_k, Ω) is HS-contact with primitive $-\lambda = -(\lambda_0 + \frac{s}{2kK}\psi)$ and the magnetic flow on S_k is contact with contact form $\frac{K}{2kK+s^2}\lambda$. If, in addition, $2kK + s^2 > 0$, then (S_k, Ω) is not Anosov. On the other hand, if $2kK + s^2 < 0$, then (S_k, Ω) is Anosov and the Anosov splitting of the magnetic flow on S_k is*

$$\begin{aligned} E^s &= \operatorname{span}\left\langle \sqrt{-(2kK + s^2)}H - V_{s,k} \right\rangle, \\ E^u &= \operatorname{span}\left\langle \sqrt{-(2kK + s^2)}H + V_{s,k} \right\rangle. \end{aligned}$$

Proof. Note that $\iota_F\lambda = 2k + \frac{s^2}{K}$. Hence, λ is a contact form if $2kK + s^2 \neq 0$, which recovers the condition $K + s'^2 \neq 0$ with the relation $s' = \frac{s}{\sqrt{2k}}$. For the Anosov condition, we do not need to redo the calculations as such is preserved by the conjugacy h and the constant time-change. Set $s' = \frac{s}{\sqrt{2k}}$. By theorem 2.6, if $K + s'^2 > 0$, then the magnetic flow does not admit an Anosov splitting. Write $(x, u) = h^{-1}(x, v)$, where $(x, v) \in S_k$. If $K + s'^2 < 0$, then the magnetic flow is Anosov and the splitting components are

$$\begin{aligned} E^s_{(x,v)} &= \operatorname{span}\left\langle (dh)_{(x,u)}\left(\sqrt{-(K+s'^2)}H(x,u)) - V_{s'}(x,u)\right)\right\rangle, \\ &= \operatorname{span}\left\langle \frac{\sqrt{-(2kK+s^2)}}{\sqrt{2k}}\frac{1}{\sqrt{2k}}H(x,v) - \frac{1}{2k}V_{s,k}(x,v)\right\rangle \end{aligned}$$

and likewise for the unstable splitting component. □

2.3 A Transversality Result for Anosov Magnetic Flows

From here on, we no longer assume the curvature or the magnetic magnitude to be constant. We will prove the transversality result we used in the previous chapter:

Theorem 2.10. *If the magnetic flow is Anosov, then at no point $(x, u) \in U\Sigma$ is the vertical vector $V(x, u) = (0, Ju)$ contained in either the weak stable or the weak unstable subspace.*

We will need the classification theorem of Anosov flows on circle bundles in dimension three. This is not restricted to magnetic flows. For a proof, see [Ghy84, Thrm. A].

Theorem 2.11. *Suppose $\phi_t \colon M \to M$ is an Anosov flow on a closed 3-manifold M that is a circle bundle. Then there exists a closed surface Σ (of genus at least 2) such that M is a finite cover of the unit tangent bundle $U\Sigma$, and ϕ_t is orbit equivalent to the lift of the geodesic flow on $U\Sigma$ to M, where the geodesic flow is understood to come from the Riemannian metric of constant curvature -1.*

Corollary 2.12. *Suppose $\phi_t \colon M \to M$ is an Anosov flow on a closed 3-manifold M that is a circle bundle. Then the flow is transitive.*

Proof. This is immediate from corollary 1.27. □

We can deduce another consequence about the existence of closed orbits of such flows. However, we first need a lemma from algebraic topology.

Lemma 2.13. *Given a circle bundle $\pi \colon S\Sigma \to \Sigma$, the induced map $\pi_* \colon H_1(S\Sigma, \mathbb{Z}) \to H_1(\Sigma, \mathbb{Z})$ has kernel the torsion subgroup of $H_1(S\Sigma, \mathbb{Z})$.*

Proof. The Gysin sequence for the circle bundle $\pi \colon S\Sigma \to \Sigma$ reads

$$\cdots \to H_2(\Sigma, \mathbb{Z}) \xrightarrow{\cap\chi} H_0(\Sigma, \mathbb{Z}) \to H_1(S\Sigma, \mathbb{Z}) \xrightarrow{\pi_*} H_1(\Sigma, \mathbb{Z}) \to 0,$$

where χ denotes the Euler class of the bundle. Since $H_2(\Sigma, \mathbb{Z}) \cong \mathbb{Z} \cong H_0(\Sigma, \mathbb{Z})$ and the map $\cap\chi$ is multiplication with the euler characteristic $2 - 2g$, we find that $H_1(S\Sigma, \mathbb{Z}) \cong \mathbb{Z}^{2g} \oplus \mathbb{Z}_{2g-2}$. □

Corollary 2.14. *Suppose $\phi_t \colon S\Sigma \to S\Sigma$ is an Anosov flow on a circle bundle $\pi \colon S\Sigma \to \Sigma$. Then any homology class in $H_1(S\Sigma, \mathbb{Z})/\ker(\pi_*)$ contains a closed orbit of ϕ_t.*

Proof. Let $\Phi \colon S\Sigma \to U\Sigma$ denote the orbit equivalence between the given flow and the geodesic flow coming from constant curvature -1 provided by theorem 2.11. Consider the induced map on homology

$\Phi_*\colon H_1(S\Sigma,\mathbb{Z}) \to H_1(U\Sigma,\mathbb{Z})$. Being an isomorphism, it must map torsion subgroups into torsion subgroups but cannot map infinite subgroups into torsion subgroups. Thus, Φ factors to an isomorphism of the free parts,

$$\Phi'\colon H_1(S\Sigma,\mathbb{Z})/\ker(\pi_*) \to H_1(U\Sigma,\mathbb{Z})/\ker(\pi^0_*),$$

where π^0 denotes the projection of the unit tangent bundle. Now given $c \in H_1(S\Sigma,\mathbb{Z})/\ker(\pi_*)$, consider $(\pi^0_* \circ \Phi')(c) \in H_1(\Sigma,\mathbb{Z})$. This homology class is known to contain a closed geodesic γ. The loop $\gamma_m(t) = \Phi^{-1}(\gamma(t),\dot\gamma(t))$ is then a closed orbit of the flow ϕ_t. In conclusion, we find that the homology class c has the closed orbit γ_m as representative. $\square$

Let us now return to magnetic flows and tend to proving theorem 2.10. For the first bit, we do not require the Anosov property. For $(x,u) \in U\Sigma$, consider the set of planes containing F, i.e.

$$\Lambda_{(x,u)}(U\Sigma) = \left\{ W \subset T_{(x,u)}U\Sigma \;\middle|\; \dim(W) = 2,\ F(x,u) \in W \right\}.$$

Denote the disjoint union over (x,u) of all these sets by $\Lambda(U\Sigma)$. Each $\Lambda_{(x,u)}(U\Sigma)$ is diffeomorphic to a circle and, in fact, $\Lambda(U\Sigma)$ admits the structure of a trivial circle bundle over $U\Sigma$. Indeed, a global trivialization is given by $W \mapsto ((x,u),\theta_{(x,u)}(W))$, where (x,u) is the base-point in whose tangent space W lives and $\theta_{(x,u)}$ is the map that describes the plane W by the angle between W and the H,V-plane. More precisely,

$$\theta_{(x,u)}(W) = \exp\left(2i \cdot \mathrm{angle}(W;\mathrm{span}(H,V))\right),$$

where the angle is measured in $[0,\pi)$. We will pay special attention to the sections $\mathcal{H}$ and $\mathcal{V}$ of the bundle given by $\mathrm{span}\langle F,H\rangle$ and $\mathrm{span}\langle F,V\rangle$, respectively. Abbreviate $\Lambda_H = \mathcal{H}(U\Sigma)$ and $\Lambda_V = \mathcal{V}(U\Sigma)$. Given an element $W \in \Lambda_{(x,u)}(U\Sigma)$ not containing $H(x,u)$, there is some real number $m(W)$ with

$$W = \mathrm{span}\langle F(x,u), m(W)H(x,u) + V(x,u)\rangle.$$

This defines a smooth map m from $\Lambda(U\Sigma) \setminus \Lambda_H$ into the real line. Observe that $\Lambda_V = m^{-1}(0)$. Now fix any point $p_0 = (x_0,u_0) \in U\Sigma$ and define $m_0(t) = m\left(d\phi_t(\mathcal{V}(p_0))\right)$.

Lemma 2.15. *It holds that $\dot m_0(0) = 1$.*

Proof. Similarly to before, we abbreviate $F(t) = F(\phi_t(p_0))$ as well as $F = F(0)$ and likewise for H and V. By definition, $m_0(t)$ is specified by

$$m_0(t)H(t) + V(t) \in d\phi_t(\mathcal{V}(p_0)) = \mathrm{span}\langle d\phi_t(F), d\phi_t(V)\rangle.$$

Thus, there are some functions a and z so that

$$m_0(t)H(t) + V(t) = a(t)d\phi_t(F) + z(t)d\phi_t(V).$$

Applying $d\phi_{-t}$ on both sides and differentiating with respect to time afterwards yields

$$\dot m_0(t)d\phi_{-t}(H(t)) + m_0(t)d\phi_{-t}([F,H](t)) + d\phi_{-t}([F,V](t)) = \dot a(t)F + \dot z(t)V.$$

We can evaluate at $t = 0$ and use $m_0(0) = 0$ to find

$$\dot m_0(0)H + [F,V] = \dot a(0)F + \dot z(0)V.$$

The Lie bracket can be calculated as

$$[F,V] = [G,V] + [sV,V] = -H - ds(V)V.$$

As F,H,V form a basis, we can extract the H component of the second-last equation to conclude $\dot m_0(0) - 1 = 0$. $\square$

There is an induced flow on $\Lambda(U\Sigma)$ given by $(t, W) \mapsto d\phi_t(W)$. Let F^* denote its infinitesimal generator. The lemma says that $(dm)_{\mathcal{V}(p_0)}$ maps $F^*(\mathcal{V}(p_0))$ to 1. Another consequence of the above lemma is that 0 is a regular value of m. In particular, we can equip $\Lambda_V = m^{-1}(0)$ with the submanifold structure from the regular value theorem. Combining the two assertions

$$\ker(dm)_{\mathcal{V}(p_0)} = T_{p_0}\Lambda_V \quad \text{and} \quad (dm)_{\mathcal{V}(p_0)}(F^*(\mathcal{V}(p_0))) = 1,$$

we find that Λ_V is orientable as a submanifold of $\Lambda(U\Sigma)$. As such, it defines a homology class in $H_3(\Lambda(U\Sigma), \mathbb{Z})$ by means of its fundamental class. By duality, there is a dual cohomology class $\mathfrak{m}$ in $H^1(\Lambda(U\Sigma), \mathbb{Z})$.

Lemma 2.16. *If a homology class $c \in H_1(\Lambda(U\Sigma), \mathbb{Z})$ can be represented by a submanifold*[2]*, then $\mathfrak{m}(c)$ is the intersection index of the submanifolds Λ_V and c.*

Proof. Abbreviate $M = \Lambda(U\Sigma)$ and $N = \Lambda_V$ and let $[M]$ and $[N]$ denote the fundamental classes. Let

$$\Psi \colon H^*_{\mathrm{dR}}(M) \to H^*(M, \mathbb{R}) \cong \mathrm{Hom}(H_*(M, \mathbb{Z}), \mathbb{R}), \; \Psi([\omega]) = \left(\sigma \mapsto \int_\sigma \omega\right)$$

denote the deRham map. It is a classical result that this is an isomorphism, which, moreover, maps wedge products to cup products. We wish to show that $c \mapsto \mathfrak{m}(c) - N \cdot c$ is the zero map in $\mathrm{Hom}(H_1(M, \mathbb{Z}), \mathbb{Z})$. There is an obvious inclusion $\mathrm{Hom}(H_1(M, \mathbb{Z}), \mathbb{Z}) \hookrightarrow \mathrm{Hom}(H_1(M, \mathbb{Z}), \mathbb{R})$. Of course, it suffices to show that the equality $\mathfrak{m}(c) - N \cdot c \equiv 0$ holds in $H^1(M, \mathbb{R})$. That $\mathfrak{m}$ is the Poincaré dual of $[N]$ means $\mathfrak{m} \cap [M] = [N]$. Let τ be a closed 1-form with $\Psi([\tau]) = \mathfrak{m}$. Take an arbitrary closed 3-form ω and set $\alpha = \Psi([\omega])$. Using

$$\Psi([\omega])([N]) = \alpha([N]) = \alpha(\mathfrak{m} \cap [M]) = (\mathfrak{m} \cup \alpha)([M]) = \Psi([\tau \wedge \omega])([M]),$$

we find that $\int_N \omega = \int_M \tau \wedge \omega$, i.e. τ is the Poincaré dual of the submanifold N in the deRham sense. As a result from Differential Geometry,

$$\mathfrak{m}(c) = \Psi([\tau])(c) = \int_c \tau = N \cdot c.$$

$\square$

Regard the weak stable subbundle E^{sc} as a section of the circle bundle $\Lambda(U\Sigma) \to U\Sigma$. Everything we do from here on can be done likewise for the weak unstable subbundle. Being a section, E^{sc} gives rise to a cohomology class $\nu = (E^{sc})^*\mathfrak{m}$ in $H^1(U\Sigma, \mathbb{Z})$. In particular, given a closed curve γ, $\nu([\gamma])$ is the intersection index in $\Lambda(U\Sigma)$ of Λ_V and $E^{sc}_{\gamma(S^1)}$. Now suppose γ is an arc of a ϕ_t-orbit and that Λ_V and $E^{sc}_{\gamma(I)}$ intersect at the point (x, u, W), $(x, u) = \gamma(0)$. This means that $E^{sc}_{(x,u)} = W = \mathcal{V}(x, u)$. As $E^{sc}_{\gamma(t)} = d\phi_t\big(E^{sc}_{(x,u)}\big)$, the intersection index is exactly

$$\Lambda_V \cdot E^{sc}_{\gamma(I)} = \frac{d}{dt}\Big|_{t=0} m\left(d\phi_t(\mathcal{V}(x, u))\right) = 1.$$

In particular, for any closed orbit γ of ϕ_t we have $\nu([\gamma]) \geq 0$. Now we invoke the Anosov condition.

Lemma 2.17. *If the magnetic flow is Anosov, then $\nu = 0$ in $H^1(U\Sigma, \mathbb{Z})$.*

Proof. Since $\nu \in \mathrm{Hom}(H_1(U\Sigma, \mathbb{Z}), \mathbb{Z})$ must vanish on any torsion elements, it factors to define a map $\nu' \colon H_1(U\Sigma, \mathbb{Z})/\ker(\pi_*)$ and it suffices to show $\nu' = 0$. However, this is immediate from corollary 2.14 and the observation that ν is positive on closed orbits of ϕ_t. $\square$

[2] It is a (non-trivial) fact that this is always the case in manifolds of dimension at least three ([Gei08, Prop. 3.4.2]).

With all these preliminaries out of the way, we can finally prove the transversality result.

Proof of theorem 2.10. As always, we only write down the proof for the weak stable subspaces. Suppose for contradiction that at some point $p_0 = (x_0, u_0)$ we have $V(x_0, u_0) \in E^{sc}_{(x_0,u_0)}$. Since $m_0(0) = 0$ and $\dot{m}_0(0) \neq 0$ by lemma 2.15, there is some ϵ-neighborhood of 0 with $m_0(t) \neq 0$ for $t \in (-\epsilon, \epsilon) \setminus \{0\}$, i.e.

$$V(\phi_t(p_0)) \notin d\phi_t(\mathcal{V}(p_0)) = d\phi_t(E^{sc}_{p_0}) = E^{sc}_{\phi_t(p_0)}.$$

As this is an open condition, there are open neighborhoods $U_\pm$ of the points $p_\pm = \phi_{\pm\epsilon/2}(p_0)$ in which the property remains true. Abbreviate $U = \phi_\epsilon(U_-) \cap U_+$. By corollary 2.12, p_+ is in the non-wandering set and, hence, there is some positive time $T' > \epsilon$ with $\phi_{T'}(U) \cap U \neq \emptyset$, i.e. we can find a point $p \in U$ with $\phi_{T'}(p) \in U$. Setting $T = T' - \epsilon > 0$, we obtain $\phi_T(p) \in \phi_{-\epsilon}(U) \subset U_-$. Now let γ denote the closed path constructed as follows: start at p_0 and run to p_+ along the orbit of p_0; then run to p without leaving U_+; after that run to $\phi_T(p)$ along the orbit; then run to p_- without leaving U_-; finally, run back to p_0 along its orbit. Now, γ defines a closed curve with $\nu([\gamma]) \geq 1$ contradicting the previous lemma. Indeed, Λ_V intersects $E^{sc}_{\gamma(S^1)}$ at p_0 by hypothesis, and since γ is an arc of a ϕ_t-orbit near p_0, the intersection index there is 1. Moreover, inside U_+ and U_- there are no intersection points by choice of $U_\pm$, and on the remaining pieces of γ we run along an orbit in forward direction, so the contribution is non-negative. □

2.4 Contact Anosov Magnetic Flows

As in the first chapter, let us add the contact property on top of the Anosov property. We mentioned the following result in remark 2.7 but postponed its proof, which we will fill in next.

Theorem 2.18. *If the magnetic flow on the unit tangent bundle is both Anosov and contact, then the magnetic magnitude s must be constant. Further, if s is not constantly zero, then the curvature is constant, as well.*

Observe that we necessarily need to exclude $s \equiv 0$ to conclude that the curvature is also constant because geodesic flows on manifolds of non-constant but strictly negative curvature provide counter-examples. We remark that this theorem subsumes the result obtained in [Pat97, p. 872], namely that σ cannot be exact in the contact Anosov case. We will deduce this theorem from the following result, which can be found in [DP05, Thrm. B]:

Theorem 2.19. *Suppose the magnetic flow on $U\Sigma$ is Anosov. Let f be a smooth function and ν be a smooth 1-form on $U\Sigma$. If there exists a smooth function g on $U\Sigma$ with $f(x) + \nu_x(u) = dg_{(x,u)}(F(x,u))$ for any $(x,u) \in U\Sigma$, then f is constantly zero and ν is exact.*

Before we can prove theorem 2.18, we first need another result from algebraic topology.

Lemma 2.20. *The projection $\pi\colon U\Sigma \to \Sigma$ induces an isomorphism $\pi^*\colon H^1(\Sigma,\mathbb{R}) \to H^1(U\Sigma,\mathbb{R})$ by pullback.*

Proof. By the Universal Coefficients Theorem, it suffices to work with coefficients in $\mathbb{Z}$. The Gysin sequence for the bundle $\pi\colon U\Sigma \to \Sigma$ reads

$$0 \to H^1(\Sigma,\mathbb{Z}) \xrightarrow{\pi^*} H^1(U\Sigma,\mathbb{Z}) \to H^0(\Sigma,\mathbb{Z}) \xrightarrow{\cup\chi} H^2(\Sigma,\mathbb{Z}) \to \cdots,$$

where χ denotes the Euler class of the bundle. From this sequence, we find that π^* is an isomorphism if and only if the map $\cup\chi$ is injective. Since $H^0(\Sigma,\mathbb{Z}) \cong \mathbb{Z}$, this amounts to χ being non-zero in $H^2(\Sigma,\mathbb{Z})$. The latter is given because we excluded the torus case. □

The second step in the proof of theorem 2.18 is the following lemma:

Lemma 2.21. *Suppose a time-change of the magnetic flow is both contact and Anosov with contact form α. Then $d\alpha = c\Omega$ for some non-zero constant $c \in \mathbb{R}$.*

Proof. We already exploited an argument like this several times in the first chapter. Since $\alpha \wedge \Omega$ is a volume form, there is a function c with $\alpha \wedge d\alpha = c\alpha \wedge \Omega$. Then c is ϕ_t-invariant since both $\alpha \wedge d\alpha$ and $\alpha \wedge \Omega$ are. The flow is transitive (corollary 2.12), so c must be constant. We can conclude the statement by contracting with the infinitesimal generator F' of the time-change:

$$d\alpha = \iota_{F'}(\alpha \wedge d\alpha) = c\iota_{F'}(\alpha \wedge \Omega) = c\Omega.$$

□

Proof of theorem 2.18. Suppose α is the contact form on $U\Sigma$ for which the magnetic flow is the Reeb flow. By the previous lemma, $d\alpha = c\Omega$ for some $c \neq 0$. Recall that $\sigma = \kappa K \Omega_{\text{area}} + d\eta$ for a real number κ and a 1-form η on Σ and, hence, $-\Omega$ admits the primitive $\lambda = \lambda_0 + \kappa\psi - \pi^*\eta$. Then $\alpha + c\lambda$ is a closed 1-form. By the algebraic topology lemma, there is a closed 1-form ρ and a smooth function g' on Σ with $\alpha + c\lambda = \pi^*\rho + d\pi^* g'$. Now set $g = g' \circ \pi$, $f = 1 + c + c\kappa s$, and $\nu = -(c\eta + \rho)$, and let us verify that these satisfy the hypothesis of theorem 2.19. Indeed,

$$dg(F) = (\alpha + c\lambda - \pi^*\rho)(F) = \alpha(F) + c\lambda_0(F) + c\kappa\psi(F) - c\pi^*\eta(F) - \pi^*\rho(F) = f + \nu.$$

Thus, f is constantly zero and ν is exact. It follows that η is closed and that s is constant or κ is zero. In particular, $s\Omega_{\text{area}} = \sigma = \kappa K \Omega_{\text{area}}$ so that $s = \kappa K$. Therefore, even if κ is zero we conclude that s is constant. Moreover, $s = \kappa K$ implies that K must be constant, as well, when $s \neq 0$. □

Together with theorem 2.9 we get the following corollary:

Corollary 2.22. *Suppose the magnetic form σ is not zero. Then the magnetic flow on the energy level S_k is both contact and Anosov if and only if both the curvature K and the magnetic magnitude s are constant and, moreover, if $2kK + s^2 < 0$.*

When the curvature is not constant, the straight-forward computations from chapter 2.2 do not go through. However, with a more elaborate use of Jacobi fields, it was proved that the Anosov condition can be recovered. The following analogue of theorem 2.9 for non-constant curvature can be found as[3] [Gou97, Thrm. 1].

Theorem 2.23. *Suppose the curvature is strictly negative with supremum $K_{\max} < 0$ and the magnetic magnitude is constant. If $2kK_{\max} + s^2 < 0$, then the magnetic flow on the energy level S_k is Anosov.*

Combining this result with theorem 2.18 yields the following corollary. We will revisit this negative-example to HS-contact Hamiltonian structures later.

Corollary 2.24. *Suppose the curvature is strictly negative but not constant with supremum $K_{\max} < 0$ and the magnetic magnitude is constant but not zero. Then the magnetic flow on the energy level S_k is Anosov but not contact for any $k > -\frac{s^2}{2K_{\max}}$. Even more so, no time-change is contact. Therefore, (S_k, Ω) is Anosov but not HS-contact.*

Proof. The first statement is an immediate application of theorems 2.18 and 2.23. When we work with a time-change, then the proof of the former goes almost through. Suppose α is the associated 1-form of the contact time-change. We again obtain a non-zero constant c, a closed 1-form ρ, and a smooth function g with $\alpha + c\lambda = \pi^*\rho + dg$. Let F' denote the infinitesimal generator of the time-change. Define ν the same way but set $f = \alpha(F') + c + c\kappa s$. Then $dg(F) = f + \nu$, as before, and theorem 2.19 tells us that f is constant. In contrast to the statement of theorem 2.18, in this corollary we assumed that s is constant. Thus, f being constant implies that $\alpha(F')$ is constant, i.e. that the time-change is trivial. □

[3] In our context, the cited theorem reduces to the version we present because for the Lorentz force Y and for any vector fields X and Z on Σ we have $(\nabla_X Y)(Z) = \nabla_X(Y(Z)) - Y(\nabla_X Z) = \nabla_X(sJZ) - sJ\nabla_X Z = (\mathcal{L}_X s)JZ$.

2.5 The Lagrangian Point of View

From here on, we will allow Σ to be a sphere or a torus. Suppose $\hat{\Sigma} \to \Sigma$ is a covering of Σ. Given any object defined on Σ or $T\Sigma$, we will denote its pullback/lift by the same letter but with a hat on top. We also pull back the Riemannian metric but this we will be implicit in the notation to keep things less messy. Assume that $\hat{\sigma} = d\theta$ is exact, where θ is a 1-form on $\hat{\Sigma}$. Consider the *Lagrangian*

$$\hat{L} \colon T\hat{\Sigma} \to \mathbb{R},\ \hat{L}(x,v) = \frac{1}{2}\langle v, v\rangle - \theta_x(v).$$

A path $x(t)$ in $\hat{\Sigma}$ is said to satisfy the *Euler-Lagrange equation* associated to $\hat{L}$ if

$$\frac{d}{dt}\frac{\partial \hat{L}}{\partial v}(x(t), \dot{x}(t)) = \frac{\partial \hat{L}}{\partial x}(x(t), \dot{x}(t)).$$

Let $\hat{Y} \colon T\hat{\Sigma} \to T\hat{\Sigma}$ denote the Lorentz force specified by $\langle \hat{Y}_x(v), w\rangle = d\theta_x(v,w)$. Indeed, this is the lift of the old Lorentz force Y on $T\Sigma$. We can use the Lorentz force to describe solutions of the Euler-Lagrange equation.

Lemma 2.25. *A path $x(t)$ in $\hat{\Sigma}$ satisfies the Euler-Lagrange equation if and only if it satisfies*

$$\nabla_t \dot{x}(t) = \hat{Y}_{x(t)}(\dot{x}(t)).$$

Proof. Take the usual local coordinates (q,p) on the tangent bundle $T\hat{\Sigma} \cong T^*\hat{\Sigma}$. Suppose θ is given in local coordinates by $\sum_j \theta_j(q) dq_j$. Then the Lorentz force $\hat{Y}^0$ with respect to the standard inner product can be written as matrix multiplication with

$$\hat{Y}^0_q = (\partial_k \theta_j(q) - \partial_j \theta_k(q))_{j,k}.$$

Note that the Lorentz force with respect to the given Riemannian metric g is $\hat{Y} = G^{-1}\hat{Y}^0$, where G denotes the matrix representing g in local coordinates. For a path $q(t)$ we get the j-th vector entry

$$\begin{aligned}&\frac{d}{dt}\left(\frac{\partial}{\partial p_j}\theta_q(p)\Big|_{(q,p)=(q(t),\dot{q}(t))}\right) - \frac{\partial}{\partial q_j}\theta_q(p)\Big|_{(q,p)=(q(t),\dot{q}(t))}\\ &= \frac{d}{dt}\theta_j(q(t)) - \sum_k \partial_j\theta_k \dot{q}_k(t) = \left(\hat{Y}^0_{q(t)}\dot{q}(t)\right)_j.\end{aligned}$$

Abbreviating $G(t) = G(q(t))$, the Euler-Lagrange equation becomes in local coordinates

$$\frac{d}{dt}\left(\dot{q}(t)^T G(t)\right) = \left(\hat{Y}^0_{q(t)}\dot{q}(t)\right)^T + \frac{1}{2}\left(\frac{\partial}{\partial q}\langle p, p\rangle_q\right)^T\Big|_{(q,p)=(q(t),\dot{q}(t))},$$

which is equivalent to

$$\ddot{q}(t) + G^{-1}(t)\left(\frac{d}{dt}G(t)\right)\dot{q}(t) - \frac{1}{2}G^{-1}(t)\left(\frac{\partial}{\partial q}\langle p,p\rangle_q\right)\Big|_{(q,p)=(q(t),\dot{q}(t))} = \hat{Y}_{q(t)}\dot{q}(t).$$

To finish the proof, we use the Christoffel symbols to verify that the left hand side is exactly $\nabla_t \dot{q}(t)$. Let g_{ij} denote the matrix entries of G and g^{ij} the entries of G^{-1}. Let us first compute the j-th vector entry of the middle term:

$$\begin{aligned}&\sum_i g^{ji}(t)\sum_k\left(\frac{d}{dt}g_{i,k}(t)\right)\dot{q}_k(t) = \sum_i g^{ji}(t)\sum_k \dot{q}_k(t)\sum_l \dot{q}_l(t)\frac{\partial}{\partial q_l}g_{i,k}(t)\\ &= \sum_{k,l}\dot{q}_k(t)\dot{q}_l(t)\sum_i \frac{1}{2}g^{ji}(t)\left(\frac{\partial}{\partial q_l}g_{i,k}(t) + \frac{\partial}{\partial q_k}g_{i,l}(t)\right).\end{aligned}$$

Then the j-th vector entry of the entire term can be written as

$$
\begin{aligned}
&\ddot{q}_j(t) + \sum_{k,l} \dot{q}_k(t)\dot{q}_l(t) \sum_i \frac{1}{2} g^{ji}(t) \left(\frac{\partial}{\partial q_l} g_{i,k}(t) + \frac{\partial}{\partial q_k} g_{i,l}(t) - \frac{\partial}{\partial q_i} g_{k,l}(t) \right) \\
=&\ddot{q}_j(t) + \sum_{k,l} \dot{q}_k(t)\dot{q}_l(t)\Gamma^j_{k,l} = \left(\nabla_t \dot{q}(t)\right)_j .
\end{aligned}
$$

□

Recall that the infinitesimal generator of the magnetic flow on Σ was also given in terms of the Lorentz force by $(u, Y_x(u))$. Thus, the last lemma states that the projection of the lifted magnetic flow to $\hat{\Sigma}$ is exactly the flow induced by the Euler-Lagrange equation. This allows us to investigate the same flow from two different point of views. There is yet another approach: instead of twisting the symplectic form we can alter the Hamiltonian. Set

$$
\hat{H} \colon T\hat{\Sigma} \to \mathbb{R},\ \hat{H}(x,v) = \frac{1}{2}\langle v + \theta_x, v + \theta_x \rangle.
$$

Then $\hat{H} = \hat{E} \circ \hat{\mathcal{L}}^{-1}$, where $\hat{\mathcal{L}}$ denotes the *Legendre transform*

$$
\hat{\mathcal{L}} \colon T\hat{\Sigma} \to T\hat{\Sigma},\ \hat{\mathcal{L}}(x,v) = \frac{\partial \hat{L}}{\partial v}(x,v) = (x, v - \theta_x).
$$

The derivatives of $\hat{H}$ and $\hat{\mathcal{L}}$ are given by

$$
\begin{aligned}
(d\hat{H})_{(x,v)}(X) &= \langle v + \theta_x, X_V + (\nabla_{X_H}\theta)(x) \rangle, \\
(d\hat{\mathcal{L}})_{(x,v)}(X) &= (X_H, X_V - (\nabla_{X_H}\theta)(x)) .
\end{aligned}
$$

Noteworthy is the following geometric observation, which will be useful later.

Lemma 2.26. *The Legendre transform $\hat{\mathcal{L}}$ is a symplectomorphism $(T\hat{\Sigma}, \hat{\omega}) \to (T\hat{\Sigma}, \hat{\omega}_0)$.*

Proof. By the above formula for the derivative of the Legendre transform, we get

$$
\begin{aligned}
(\hat{\mathcal{L}}^* \hat{\omega}_0)(X,Y) &= \hat{\omega}_0(X,Y) + \langle Y_H, \nabla_{X_H}\theta \rangle - \langle X_H, \nabla_{Y_H}\theta \rangle \\
&= \hat{\omega}_0(X,Y) + \langle Y_H, \nabla_{X_H}\theta \rangle - \langle X_H, \nabla_{Y_H}\theta \rangle + \langle \underbrace{\nabla_{X_H} Y_H - \nabla_{Y_H, X_H} + [X_H, Y_H]}_{=0}, \theta \rangle \\
&= \hat{\omega}_0(X,Y) + \mathcal{L}_{X_H}(\theta(Y_H)) - \mathcal{L}_{Y_H}(\theta(X_H)) + \theta([X_H, Y_H]) \\
&= \hat{\omega}_0(X,Y) + \pi^* d\theta(X,Y) = \hat{\omega}(X,Y).
\end{aligned}
$$

□

The Legendre transform also carries dynamical meaning. Though, to exploit it, we first need to do some more calculations.

Lemma 2.27. *For any vectors $u, w \in T\hat{\Sigma}$, it holds that*

$$
\langle u, \nabla_w \theta \rangle = \langle \nabla_u \theta, w \rangle - \langle Y(u), w \rangle.
$$

Proof. Let W be a vector field with $W(x) = w$ and let ρ_t denote its flow. Take a path $\gamma(s)$ with $\gamma(0) = x$ and $\dot{\gamma}(0) = u$. Note that

$$
\big(\nabla_t d\rho_t(u)\big)(0) = \big(\nabla_t \partial_s(\rho_t \circ \gamma)\big)(0)(0) = \big(\nabla_s \partial_t(\rho_t \circ \gamma)\big)(0)(0) = \big(\nabla_u W\big)(x).
$$

In addition, we observe $(\iota_W d\theta)_x(u) = -\langle Y(u), w\rangle$ and compute further

$$(d\iota_W\theta)_x(u) = \frac{d}{ds}\Big|_{s=0}\langle\theta_{\gamma(s)}, W(\gamma(s))\rangle = \langle\nabla_u\theta, w\rangle + \langle\theta_x, \nabla_u W\rangle$$

Combining these equalities with Cartan's formula yields

$$\begin{aligned}\langle\nabla_u\theta, w\rangle + \langle\theta_x, \nabla_u W\rangle - \langle Y(u), w\rangle &= (d\iota_W\theta)_x(u) + (\iota_W d\theta)_x(u)\\ &= (\mathcal{L}_W\theta)_x(u) = \frac{d}{dt}\Big|_{t=0}\theta_{\rho_t(x)}(d\rho_t(u))\\ &= \frac{d}{dt}\Big|_{t=0}\langle\theta_{\rho_t(x)}, d\rho_t(u)\rangle = \langle\nabla_w\theta, u\rangle + \langle\theta_x, \nabla_u W\rangle.\end{aligned}$$

□

We can consider the symplectic gradient $F_{\hat{H}}$ of $\hat{H}$ with respect to the standard form $\hat{\omega}_0$, i.e. the unique vector field satisfying $\iota_{F_{\hat{H}}}\hat{\omega}_0 = d\hat{H}$. Then the flow generated by $F_{\hat{H}}$ is exactly the conjugate of the magnetic flow under the Legendre transform,

$$F_{\hat{H}}(x, v) = \frac{d}{dt}\Big|_{t=0}\hat{\mathcal{L}}\circ\hat{\phi}_t\circ\hat{\mathcal{L}}^{-1}(x, v) = \big(v + \theta_x, Y(v + \theta_x) - (\nabla_{(v+\theta_x)}\theta)(x)\big)\,.$$

Indeed, with the short-hands $u = v + \theta_x$ and $w = X_H$, the lemma shows

$$d\hat{H}_{(x,v)}(X) = \langle u, X_V + \nabla_w\theta\rangle = \langle u, X_V\rangle + \langle\nabla_u\theta, w\rangle - \langle Y(u), w\rangle = \hat{\omega}_0(F_{\hat{H}}, X).$$

In other words, if $\phi_t^{\hat{H}}$ denotes the symplectic gradient flow of $\hat{H}$ with respect to $\hat{\omega}_0$, then $\hat{\mathcal{L}}\circ\hat{\phi}_t = \phi_t^{\hat{H}}\circ\hat{\mathcal{L}}$. Moreover, since $\hat{H}^{-1}(k) = \hat{\mathcal{L}}(\hat{E}^{-1}(k)) = \hat{\mathcal{L}}(\hat{S}_k)$, the restriction of $\hat{\phi}_t$ to $\hat{S}_k$ and the restriction of $\phi_t^{\hat{H}}$ to $\hat{H}^{-1}(k)$ are also conjugated by $\hat{\mathcal{L}}$. Lastly, observe that

$$\hat{H}(x, v) = \langle v, \hat{\mathcal{L}}(x, v)\rangle - \hat{L}(x, \hat{\mathcal{L}}(x, v)) = \max_{w\in T_x\hat{\Sigma}}\langle v, w\rangle - \hat{L}(x, w) = \hat{L}^*(x, v),$$

i.e. the new Hamiltonian is exactly the dual function of the Lagrangian.

2.6 Mañé's Critical Values

In this section, we will fix the magnetic form σ and consider variations of the energy level. Using the Lagrangian point of view, we can consider the *action of* $\hat{L}$

$$A_{\hat{L}}\colon \mathcal{AC}\to\mathbb{R},\ A_{\hat{L}}(\gamma) = \int_0^T \hat{L}(\gamma(t), \dot{\gamma}(t))dt,$$

where $\gamma\colon [0, T]\to\hat{\Sigma}$ is an element of the set $\mathcal{AC} = \mathcal{AC}(\hat{\Sigma})$ of absolutely continuous curves in $\hat{\Sigma}$. Absolute continuity is the appropriate level of regularity to impose because it ensures that the derivative of γ exists Lebesque-almost everywhere and that the integrand is Lebesque-integrable. Define the *Mañé critical value of the Lagrangian* $\hat{L}$ as

$$\begin{aligned}c(\hat{L}) &= \inf\big\{k\in\mathbb{R}\ \big|\ \forall \text{ closed } \gamma\in\mathcal{AC}\colon A_{\hat{L}+k}(\gamma)\geq 0\big\}\\ &= \sup\big\{k\in\mathbb{R}\ \big|\ \exists \text{ closed } \gamma\in\mathcal{AC}\colon A_{\hat{L}+k}(\gamma) < 0\big\}.\end{aligned}$$

The set in the second line is clearly an open set, so the supremum is never a maximum. Conversely, the infimum is always a minimum (unless $c(\hat{L}) = \infty$). Note that if we take a constant curve, then $A_{\hat{L}+k}(\gamma) = kT < 0$ for $k < 0$. Thus, we necessarily have $c(\hat{L})\geq 0$.

Remark 2.28. *Suppose* $\Pi\colon \bar{\Sigma} \to \hat{\Sigma}$ *is a cover. Since* $A_{\bar{L}+k}(\gamma) = A_{\hat{L}+k}(\Pi \circ \gamma)$ *for any closed absolutely continuous curve* γ *in* $\bar{\Sigma}$*, we find*

$$c(\bar{L}) = \sup\left\{k \in \mathbb{R} \;\middle|\; \exists \text{ closed } \gamma \in \mathcal{AC}(\bar{\Sigma})\colon A_{\hat{L}+k}(\Pi \circ \gamma) < 0\right\} \le c(\hat{L}).$$

In general, this inequality could be strict. However, if $\Pi\colon \bar{\Sigma} \to \hat{\Sigma}$ *is a finite cover, then* $c(\bar{L}) = c(\hat{L})$.

Proof. Indeed, if $\gamma \in \mathcal{AC}(\hat{\Sigma})$ satisfies $A_{\hat{L}+k}(\gamma) < 0$, then for any lift $\bar{\gamma}$ to $\bar{\Sigma}$ we have $A_{\bar{L}+k}(\bar{\gamma}) < 0$, but $\bar{\gamma}$ may not be closed. If the cover is finite, then concatenating finitely many lifts, we eventually obtain a closed curve with negative action (since the action is additive with respect to concatenation) and, hence, $c(\bar{L}) \ge c(\hat{L})$. □

Let $\mathcal{AC}(x, y)$ denote the subset of $\mathcal{AC}$ of curves that start at x and end at y. Then, we can define the *action potential*

$$\Phi_k(x, y) = \inf_{\gamma \in \mathcal{AC}(x,y)} A_{\hat{L}+k}(\gamma) \in [-\infty, \infty).$$

Originally, the critical value was found as the unique number

$$\begin{aligned} c(\hat{L}) &= \inf\left\{k \in \mathbb{R} \;\middle|\; \forall x, y \in \hat{\Sigma}\colon \Phi_k(x, y) > -\infty\right\} \\ &= \sup\left\{k \in \mathbb{R} \;\middle|\; \forall x, y \in \hat{\Sigma}\colon \Phi_k(x, y) = -\infty\right\}, \end{aligned}$$

which is the assertion of the first part of the lemma below. As before, the supremum is never a maximum and the infimum is always a minimum (unless $c(\hat{L}) = \infty$).

Lemma 2.29. *The critical value* $c(\hat{L})$ *is the unique number in* $[0, \infty]$ *satisfying the following properties: If* $k < c(\hat{L})$*, then* $\Phi_k(x, y) = -\infty$ *for all* $x, y \in \hat{\Sigma}$*. On the other hand, if* $k \ge c(\hat{L})$*, then* $\Phi_k(x, y) > -\infty$ *for all* $x, y \in \hat{\Sigma}$*. Thus, for* $k \ge c(\hat{L})$*, it makes sense to say for all* $x, y, z \in \hat{\Sigma}$

1. $\Phi_k(x, z) \le \Phi_k(x, y) + \Phi_k(y, z)$, *(triangle inequality),*
2. $\Phi_k(x, x) = 0$,
3. $\Phi_k(x, y) + \Phi_k(y, x) \ge 0$,
4. Φ_k *is locally Lipschitz.*

If the cohomology class of θ *contains a bounded representative, then* Φ_k *even is uniformly Lipschitz and, moreover,* $\Phi_k(x, y) + \Phi_k(y, x) > 0$ *if* $x \ne y$ *and* $k > c(\hat{L})$.

Proof. The triangle inequality in the first item simply follows from taking the concatenation of a curve γ in $\mathcal{AC}(x, y)$ and a curve η in $\mathcal{AC}(y, z)$:

$$\Phi_k(x, z) \le A_{\hat{L}+k}(\gamma * \eta) = A_{\hat{L}+k}(\gamma) + A_{\hat{L}+k}(\eta).$$

Note that this makes sense even if one of the action potentials is $-\infty$. To settle the first statement about $k < c(\hat{L})$, take some $z \in \hat{\Sigma}$ and some $\gamma \in \mathcal{AC}(z, z)$ with $A_{\hat{L}+k}(\gamma) < 0$. Then, for any N

$$\Phi_k(z, z) \le A_{\hat{L}+k}(\gamma \overset{N \text{ times}}{* \cdots *} \gamma) = N A_{\hat{L}+k}(\gamma) \xrightarrow{N \to \infty} -\infty.$$

As we showed item one also for the negative infinite case, we can conclude for any $x, y \in \hat{\Sigma}$

$$\Phi_k(x, y) \le \Phi_k(x, z) + \Phi_k(z, z) + \Phi_k(z, y) = -\infty.$$

Conversely, if $\Phi_k(x, y) = -\infty$ for some $x, y \in \hat{\Sigma}$, then also

$$\Phi_k(x, x) \le \Phi_k(x, y) + \Phi_k(y, x) = -\infty.$$

Hence, by definition of $\Phi_k(x,x)$, there is a curve $\gamma \in \mathcal{AC}(x,x)$ with $A_{\hat{L}+k}(\gamma) < 0$. Then $k \leq c(\hat{L})$, but we even have a strict inequality as the supremum defining $c(\hat{L})$ is never a maximum. Next, we prove items two to four. Of course, they are trivial if $c(\hat{L}) = \infty$. Thus, fix $\infty > k \geq c(\hat{L})$. Item two is basically obvious: by definition of the critical value, we surely have $\Phi_k(x,x) \geq 0$; on the other hand, by taking a constant curve, $A_{\hat{L}+k}(\gamma) = kT$ can be made arbitrarily small. The third item is an immediate consequence of the first two. For the fourth, fix a relatively compact, open, geodesically convex, set U in $\hat{\Sigma}$. Then the term $|\hat{L}(x,v) + k|$ is bounded by some constant Q uniformly in $\{(x,v) \mid x \in U,\ |v| \leq 1\}$, by compactness. Let $T = \mathrm{dist}(x_1,x_2)$, for $x_1, x_2 \in U$, and let γ be a unit speed geodesic in U between x_1 and x_2. Then

$$\Phi_k(x_1,x_2) \leq A_{\hat{L}+k}(\gamma) \leq Q\mathrm{dist}(x_1,x_2).$$

By the triangle inequality,

$$\begin{aligned}\Phi_k(x_1,y_1) - \Phi_k(x_2,y_2) \leq \Phi_k(x_1,x_2) + \Phi_k(y_2,y_1) &\leq Q\left(\mathrm{dist}(x_1,x_2) + \mathrm{dist}(y_2,y_1)\right) \\ &= Q\mathrm{dist}((x_1,y_1),(x_2,y_2)).\end{aligned}$$

As the argument is symmetric in the given points, this proves the Lipschitz property of the action potential inside U. Now suppose that the cohomology class of θ contains a bounded representative in the sense that there exists a smooth function u on $\hat{\Sigma}$ with $\sup_{x\in\hat{\Sigma}} |\theta_x + (du)_x| < \infty$. By Stokes' theorem, the action of the Lagrangian $\hat{L}' = \hat{L} - du$ is the same as the action of $\hat{L}$, so we may have worked with $\hat{L}'$ to begin with. This reduces us to the case in which θ itself is bounded. Then $|\hat{L}(x,v) + k|$ can be bounded uniformly in $\{(x,v) \mid x \in \hat{\Sigma},\ |v| \leq 1\}$ and we conclude that Φ_k is uniformly Lipschitz in this case. Finally, assume $x \neq y$ and $k > c(\hat{L})$. Note that boundedness of θ ensures that the Lagrangian is superlinear, i.e. given any $A > 0$ there is some $B > 0$ (for instance, $B = 2(A + ||\theta||_\infty)^2$) with $\hat{L}(x,v) \geq A|v| - B$ for any $(x,v) \in T\hat{\Sigma}$. Take a sequence of curves $\gamma_n \colon [0,T_n] \to \hat{\Sigma}$ in $\mathcal{AC}(x,y)$ with $A_{\hat{L}+k}(\gamma_n) \to \Phi_k(x,y)$. The periods T_n must be uniformly bounded from below for otherwise $T_n \to 0$ on a subsequence and then

$$\Phi_k(x,y) \geq \lim_n \int_0^{T_n} A|\dot{\gamma}_n(t)|dt - BT_n + kT_n \geq A \cdot \mathrm{dist}(x,y) \xrightarrow{A\to\infty} \infty,$$

a contradiction. Take any $0 < T < \liminf_n T_n$. Similarly, we can take a sequence $\eta_n \colon [0,T'_n] \to \hat{\Sigma}$ in $\mathcal{AC}(y,x)$ with $A_{\hat{L}+k}(\eta_n) \to \Phi_k(y,x)$ and some $0 < T' < \liminf_n T'_n$. Then

$$\begin{aligned}0 \leq \Phi_{c(\hat{L})}(x,x) &\leq \lim_n A_{\hat{L}+c(\hat{L})}(\gamma_n * \eta_n) \\ &= \lim_n A_{\hat{L}+k}(\gamma_n) + (c(\hat{L}) - k)T_n + A_{\hat{L}+k}(\eta_n) + (c(\hat{L}) - k)T'_n \\ &\leq \Phi_k(x,y) + \Phi_k(y,x) + \underbrace{(c(\hat{L}) - k)(T + T')}_{<0}.\end{aligned}$$

□

The next proposition gives a more concrete description of the critical value. Note that the expression appearing in the supremum is exactly the new Hamiltonian $\hat{H}$ with input $(du)_x$.

Proposition 2.30.

$$\begin{aligned}c(\hat{L}) &= \inf_{u\in C^\infty(\hat{\Sigma},\mathbb{R})} \sup_{x\in\hat{\Sigma}} \frac{1}{2}|(du)_x + \theta_x|^2 \\ &= \inf\left\{k \in \mathbb{R} \;\middle|\; \exists u \in C^\infty(\hat{\Sigma},\mathbb{R})\colon \sup_{x\in\hat{\Sigma}} \hat{H}(x,(du)_x) < k\right\}.\end{aligned}$$

Proof. The two infima are clearly the same. We first prove that the critical value is smaller or equal to the right hand side. This is trivial if the right hand side is infinite. Thus, assume it equals some $k \in [0, \infty)$. Then the infimum is a minimum and there is some function $u \in C^\infty(\hat{\Sigma}, \mathbb{R})$ for which $\sup_x \hat{H}(x, (du)_x) = k$. Take any curve $\gamma \in \mathcal{AC}(x, x)$ for arbitrary $x \in \hat{\Sigma}$. Using that

$$(du)_x(v) - \hat{L}(x, v) \leq \hat{L}^*(x, (du)_x) = \hat{H}(x, (du)_x) \leq k$$

uniformly in v, we conclude for the action with the help of Stokes' theorem

$$A_{\hat{L}+k}(\gamma) = \int_0^T \Big(\hat{L}(\gamma(t), \dot{\gamma}(t)) + k - (du)_{\gamma(t)}(\dot{\gamma}(t)) \Big)\, dt \geq 0.$$

By definition, this implies $c(\hat{L}) \leq k$. Now suppose $c(\hat{L}) < \infty$. We want to prove that the right hand side is at most $c(\hat{L})$. Fix a point $x_0 \in \hat{\Sigma}$ and consider the (not necessarily smooth) function $u(x) = \Phi_{c(\hat{L})}(x_0, x)$. The idea is to show that $\hat{H}(x, (du)_x) \leq c(\hat{L})$ holds for almost every point and then argue that we can smoothly approximate u in a well-behaved manner with respect to our desired inequality. As the action potential is (locally) Lipschitz as shown in lemma 2.29, so is the function u. Hence, by Rademacher's theorem, u is differentiable almost everywhere. Let x be a point of differentiability for u. Take a differentiable curve γ with initial data (x, v). Note that $u(y) - u(x) \leq \Phi_{c(\hat{L})}(x, y)$ for any point y by the triangle inequality. Then

$$\begin{aligned}(du)_x(v) &= \limsup_{t \to 0} \frac{1}{t} \Big(u(\gamma(t)) - u(x) \Big) \leq \limsup_{t \to 0} \frac{1}{t} \Phi_{c(\hat{L})}(x, \gamma(t)) \\ &\leq \limsup_{t \to 0} \frac{1}{t} \int_0^t \Big(\hat{L}(\gamma(s), \dot{\gamma}(s)) + c(\hat{L}) \Big)\, ds = \hat{L}(x, v) + c(\hat{L}).\end{aligned}$$

As v was arbitrary,

$$\hat{H}(x, (du)_x) = \max_{v \in T_x \hat{\Sigma}} (du)_x(v) - L(x, v) \leq c(\hat{L}).$$

The proof is finished if we can find a smooth approximation u' of u satisfying $\sup_x \hat{H}(x, (du')_x) \leq c(\hat{L})$. To this end, embed Σ in euclidean space $\mathbb{R}^N$ and take a small tubular neighborhood U of Σ in $\mathbb{R}^N$. We can define a projection $\rho \colon U \to \Sigma$ via the normal bundle of Σ. For the smoothing, take a smooth non-negative function $\chi \colon \mathbb{R} \to \mathbb{R}$ with support in $(-\epsilon, \epsilon)$ and $\int_{\mathbb{R}^N} \chi(|y|) dy = 1$. Next, define $K \colon \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ by $K(z, y) = \chi(|z - y|^2)$. Recall that any Borel probability measure μ on Σ is uniquely determined by its associated map $C^0(\Sigma, \mathbb{R}) \to \mathbb{R}$ sending ϕ to $\int_\Sigma \phi d\mu$. Now consider the family μ_z of Borel probability measures on Σ specified by

$$\int_\Sigma \phi \; d\mu_z = \int_{\mathbb{R}^N} (\phi \circ \rho)(y) \; K(z, y) \; dy$$

for any $\phi \in C^0(\Sigma, \mathbb{R})$. Note that the integral on the right hand side is well-defined for a sufficiently small choice of ϵ since then $K(z, y) = 0$ whenever $\rho(y)$ is not defined. We want to lift these measures to $\hat{\Sigma}$. In what follows, all balls are open and have radius ϵ. First observe that the support of μ_z is contained in the ball B_z. Let Π denote the covering map of $\hat{\Sigma} \to \Sigma$. For small ϵ, the restriction of Π to the ball B_x, $x \in \hat{\Sigma}$, is a diffeomorphism onto $B_{\Pi(x)}$. Thus, we can define $\hat{\mu}_x = \big(\Pi|_{B_x}^{-1}\big)_* \mu_{\Pi(x)}$. Define the approximation $u'(x) = \int_{\hat{\Sigma}} u \; d\hat{\mu}_x$. On any very small open set V in $\hat{\Sigma}$, we can restrict Π suitably to a set which contains all B_x, $x \in V$, (indicated by $\Pi|$) so that $\Pi|$ is still a diffeomorphism onto its image and we can write

$$u'(x) = \int_{\mathbb{R}^N} (u \circ \Pi|^{-1} \circ \rho)(y) \; K(\Pi(x), y) \; dy.$$

From this expression we find that u' is smooth because K is. It remains to verify $\sup_x \hat{H}(x,(du')_x) \leq k$. We begin by computing du'. We already observed that u is differentiable almost everywhere, which implies that u is weakly differentiable with weak derivatives given by the almost everywhere defined du. Let x_1, x_2 be a coordinate system on $\hat{\Sigma}$ and take any coordinate system $y_j = z_j$, $1 \leq j \leq N$, on $\mathbb{R}^N$ with $d\rho(\frac{\partial}{\partial y_j}) = d\Pi(\frac{\partial}{\partial x_j})$ for $j = 1, 2$. This way,

$$\frac{\partial}{\partial y_j}(u \circ \Pi|^{-1} \circ \rho)(y) = (du)_{\Pi|^{-1}\circ\rho(y)} \circ d\Pi|^{-1} \circ d\rho\Big(\frac{\partial}{\partial y_j}\Big) = \Big(\frac{\partial}{\partial x_j} u\Big) \circ (\Pi|^{-1} \circ \rho)(y)$$

Restricted to Σ, ρ is the identity, so $d\rho(\frac{\partial}{\partial y_j}) = \frac{\partial}{\partial y_j}$ on Σ. Therefore,

$$\frac{\partial}{\partial x_j} K(\Pi(x), y) = d_z K \circ d\Pi\Big(\frac{\partial}{\partial x_j}\Big) = d_z K \circ d\rho\Big(\frac{\partial}{\partial z_j}\Big) = \frac{\partial}{\partial z_j} K(z,y)\Big|_{(z,y)=(\Pi(x),y)}$$

Using $\partial_{z_j} K(z,y) = -\partial_{y_j} K(z,y)$, we compute

$$\begin{aligned}
\frac{\partial}{\partial x_j} u'(x) &= \frac{\partial}{\partial x_j} \int_{\mathbb{R}^N} (u \circ \Pi|^{-1} \circ \rho)(y)\ K(\Pi(x), y)\ dy \\
&= \int_{\mathbb{R}^N} (u \circ \Pi|^{-1} \circ \rho)(y)\ \partial_{z_j} K(z,y)|_{(z,y)=(\Pi(x),y)}\ dy \\
&= -\int_{\mathbb{R}^N} (u \circ \Pi|^{-1} \circ \rho)(y)\ \partial_{y_j} K(z,y)|_{(z,y)=(\Pi(x),y)}\ dy \\
&= \int_{\mathbb{R}^N} \partial_{y_j}(u \circ \Pi|^{-1} \circ \rho)(y)\ K(\Pi(x), y)\ dy \\
&= \int_{\mathbb{R}^N} ((\partial_{x_j} u) \circ \Pi|^{-1} \circ \rho)(y)\ K(\Pi(x), y)\ dy \\
&= \int_{\hat{\Sigma}} (\partial_{x_j} u)\ d\hat{\mu}_x.
\end{aligned}$$

Since $\hat{H}$ is convex, we conclude with Jensen's inequality $\hat{H}(x,(du')_x) \leq \sup_y \hat{H}(y,(du)_y) \leq c(\hat{L})$, where the supremum is taken over all points of differentiability for u. □

Remark 2.31 (The symplectic point of view)**.** *Recall that the graph of a 1-form on $\hat{\Sigma}$ is a Lagrangian submanifold of $(T\hat{\Sigma}, \hat{\omega}_0)$ if and only if the 1-form is closed. In particular, the graph of an exact 1-form is a Lagrangian submanifold, called an* exact Lagrangian graph*. Proposition 2.30 can be rephrased as follows: $c(\hat{L})$ is the infimum over all $k \in \mathbb{R}$ for which the sublevel set $\hat{H}^{-1}(-\infty, k)$ endowed with $\hat{\omega}_0|_{\hat{H}^{-1}(k)}$ contains an exact Lagrangian graph (namely, the graph of du).*

By a similar expression as the one in the proposition, we can define a different critical value. This one does not depend on the Lagrangian but only on the Hamiltonian (the Lagrangian depends on the choice of primitive θ and the Riemannian metric while the Hamiltonian only depends on the metric). Set

$$c(\hat{E}) = \inf_{\Theta} \sup_{x \in \hat{\Sigma}}\ \hat{E}(x, \Theta_x) = \inf_{\Theta} \sup_{x \in \hat{\Sigma}}\ \frac{1}{2}|\Theta_x|^2,$$

where the infimum is taken over all possible primitives of $\hat{\sigma}$. We call this the *Mañé critical value of the energy Hamiltonian $\hat{E}$*. By definition, $c(\hat{E})$ is finite if and only if $\hat{\sigma}$ admits some bounded primitive.

Remark 2.32. *As in remark 2.28, suppose $\Pi\colon \bar{\Sigma} \to \hat{\Sigma}$ is a cover. Since any primitive of $\hat{\sigma}$ can be pulled back to a primitive of $\bar{\sigma}$, we find*

$$c(\bar{E}) \leq \inf_{d\Theta = \hat{\sigma}} \sup_{x \in \bar{\Sigma}}\ \frac{1}{2}|(\Pi^*\Theta)_x|^2 = c(\hat{E}).$$

Again, this inequality might be strict but only for infinite covers.

The proof of this remark is a special instance of the theme of proposition 2.35.

Proof. To see this, suppose $\Pi\colon \bar{\Sigma} \to \hat{\Sigma}$ is a finite cover and Θ is a primitive of $\bar{\sigma} = \Pi^*\hat{\sigma}$. By assumption, the group Γ of deck transformations of $\bar{\Sigma} \to \hat{\Sigma}$ is finite. Set $\Theta' = \frac{1}{|\Gamma|}\sum_{g\in\Gamma} g^*\Theta$. Then Θ' descends to a form on $\hat{\Sigma}$ with the same ∞-norm as Θ since Γ acts by isometries. Moreover, Θ' is a primitive of $\hat{\sigma}$ since $dg^*\Theta = g^*\Pi^*\hat{\sigma} = \hat{\sigma}$. □

In contrast to the definition of $c(\hat{E})$, in proposition 2.30 we minimize over all possible primitives Θ for which $\Theta - \theta$ is exact. In particular, we immediately see how the two critical values are linked.

Corollary 2.33.

$$c(\hat{E}) = \inf_{[\Theta]\in H^1(\hat{\Sigma},\mathbb{R})} c(\hat{L} - \Theta).$$

If $\hat{\Sigma}$ has trivial first cohomology group, then this reduces to $c(\hat{E}) = c(\hat{L})$. This is certainly the case when we work with the universal cover $\tilde{\Sigma}$. For this, we introduce the notation $c_u = c(\tilde{E}) = c(\tilde{L})$ and call it the *universal Mañé critical value.*

Remark 2.34. *Suppose briefly that the magnetic magnitude and the curvature are constant with $K < 0$. Then the universal cover of Σ is the upper half plane $\mathbb{H}$ and the magnetic form lifts to $sdx \wedge dy$. By proposition 2.30, the universal Mañé critical value is*

$$\begin{aligned} c_u &= \inf_{u\in C^\infty(\mathbb{H},\mathbb{R})} \sup_{(x,y)\in\mathbb{H}} \frac{1}{2}\left|(du)_{(x,y)} + sxdy\right|^2_{\text{hyperbolic}} \\ &= \inf_{u\in C^\infty(\mathbb{H},\mathbb{R})} \sup_{(x,y)\in\mathbb{H}} \frac{-1}{2K}\frac{|\partial_x u|^2_{\text{eucl}} + |\partial_y u + sx|^2_{\text{eucl}}}{y^2} = -\frac{s^2}{2K}, \end{aligned}$$

where we used that a function u achieving this infimum is $u(x,y) = -sxy$. Observe that this is exactly the distinguished value appearing in theorem 2.9, respectively 2.6.

More generally, other interesting covers are those with amenable group of deck transformations:

Proposition 2.35. *Suppose $\bar{\Sigma} \to \hat{\Sigma}$ is a covering with amenable group of deck transformations Γ. Then $c(\bar{E}) = c(\hat{E})$. In particular, if $\infty > k > c(\bar{E})$, then there exists a Γ-invariant primitive Θ' on $\bar{\Sigma}$ with $\frac{1}{2}||\Theta'||^2_\infty \le k$.*

Proof. We make a short remark concerning notation: since we usually denote simplices by σ, let us denote the magnetic form σ by σ_m in this proof for increased readability. Since we already know $c(\bar{E}) \le c(\hat{E})$ and the statement is trivial if $c(\bar{E}) = \infty$, suppose $c(\bar{E}) < k < \infty$. We need to show that $c(\hat{E}) \le k$. The basic idea is that amenability enables us to average a given form on $\bar{\Sigma}$ so that it descends to $\hat{\Sigma}$. However, the technical realization is slightly more involved. Let

$$\bar{\Psi}\colon \Omega^*(\bar{\Sigma}) \to C^*(\bar{\Sigma}) \cong \mathrm{Hom}(C_*(\bar{\Sigma}),\mathbb{R}),\ \bar{\Psi}(\omega) = \left(\sigma \mapsto \int_\sigma \omega\right)$$

denote the deRham map again, which induces an isomorphism in cohomology. Amenability of the group of deck transformations Γ says that there exists a right-invariant mean, i.e. a non-negative linear functional $\mu\colon L^\infty(\Gamma) \to \mathbb{R}$ of norm one with $\mu(l\cdot g) = \mu(l)$, where $l\cdot g\colon h \mapsto l(hg)$. The assumption $k > c(\bar{E})$ implies the existence of a primitive Θ of $\bar{\sigma}_m$ with $\frac{1}{2}||\Theta||^2 > k$. Define a family of maps $l_\sigma\colon \Gamma \to \mathbb{R}$ by sending g to $\bar{\Psi}(g^*\Theta)(\sigma)$. These maps are bounded since Γ acts by isometries and, hence, $|l_\sigma(g)| \le ||\Theta||_\infty \cdot \mathrm{length}(\sigma)$.

Thus, it makes sense to define $\bar{M} \in C^*(\bar{\Sigma})$ by $\bar{M}(\sigma) = \mu(l_\sigma)$. Let us compute its boundary. Since $d\Theta = \bar{\sigma}_m$ is Γ-invariant, we obtain

$$l_{\partial\sigma}(g) = \int_{\partial\sigma} g^*\Theta = \int_\sigma g^* d\Theta = \int_\sigma \bar{\sigma}_m,$$

i.e. $l_{\partial\sigma}$ is constant in g and, therefore, $\partial\bar{M}(\sigma) = \mu(l_{\partial\sigma}) = \bar{\Psi}(\bar{\sigma}_m)(\sigma)$. Denote by $\hat{\Psi}$ the deRham map for $\hat{\Sigma}$. By definition, $\bar{M}$ is Γ-invariant as is $\bar{\sigma}_m$, so the last equation induces $\partial\hat{M} = \hat{\Psi}(\hat{\sigma}_m)$ on $C^*(\hat{\Sigma})$. Since $\hat{\Psi}$ induces an isomorphism in cohomology, there exists a smooth primitive α of $\hat{\sigma}_m$. Now $\partial(\hat{M} - \hat{\Psi}(\alpha)) = 0$, so that the cohomology class of $\hat{M} - \hat{\Psi}(\alpha)$ is the image of $[\beta]$ under $\hat{\Psi}$ for some closed 1-form β on $\hat{\Sigma}$. Denote by θ the primitive $\alpha + \beta$ of $\hat{\sigma}_m$ and by $\hat{L}$ the Lagrangian defined by θ. Then given $\gamma \in \mathcal{AC}(\hat{\Sigma})$ of length $l(\gamma)$, $\gamma\colon [0,T] \to \hat{\Sigma}$, we can use

$$\Big|\int_\gamma \theta\Big| = |\hat{\Psi}(\alpha+\beta)(\gamma)| = |\hat{M}(\gamma)| \leq ||\Theta||_\infty l(\gamma)$$

to find that the action of this Lagrangian on energy k is

$$\begin{aligned} A_{\hat{L}+k}(\gamma) &= \int_\gamma \Big(\frac{1}{2}|v|^2 - \theta + k\Big) > \frac{l(\gamma)^2}{2T} - ||\Theta||_\infty l(\gamma) + \frac{T}{2}||\Theta||_\infty^2 \\ &= \frac{T}{2}\Big(\frac{l(\gamma)}{T} - ||\Theta||_\infty\Big)^2 > 0. \end{aligned}$$

Since γ was arbitrary, we obtain $c(\hat{L}) \leq k$ by definition. Finally, corollary 2.33 yields $c(\hat{E}) \leq c(\hat{L}) \leq k$, as desired. □

Among this class of covers is the abelian cover Σ^{ab}, which has as fundamental group the commutator subgroup of $\pi_1(\Sigma, \star)$. Indeed, the group of deck transformations of this cover is exactly the abelianization of $\pi_1(\Sigma, \star)$, hence amenable.

Lemma 2.36. *For the abelian cover, the induced map $H^1(\Sigma, \mathbb{R}) \to H^1(\Sigma^{ab}, \mathbb{R})$ by pull-back is trivial.*

Proof. We will neglect the base-points of the fundamental groups in this proof. Let $h\colon \pi_1(\Sigma) \to H_1(\Sigma, \mathbb{Z})$ and $h^{ab}\colon \pi_1(\Sigma^{ab}) \to H_1(\Sigma^{ab}, \mathbb{Z})$ denote the Hurewicz maps. Then the induced map $\tilde{h}\colon \pi_1(\Sigma)^{ab} \to H_1(\Sigma, \mathbb{Z})$ from the universal property defining the abelianization is an isomorphism by Hurewicz' theorem. Let Π denote the covering map of the abelian cover. Denote by $\mathcal{C}(\pi_1(\Sigma))$ the commutator subgroup of $\pi_1(\Sigma)$ so that $\Pi_*\colon \pi_1(\Sigma^{ab}) \to \pi_1(\Sigma)$ is an isomorphism onto $\mathcal{C}(\pi_1(\Sigma))$. Then the following diagram clearly commutes:

$$\begin{tikzcd} \pi_1(\Sigma^{ab}) \arrow[r, two heads, "h^{ab}"] \arrow[d, "\Pi_*", "\wr"'] & H_1(\Sigma^{ab}, \mathbb{Z}) \arrow[rr, "\Pi_*"] & & H_1(\Sigma, \mathbb{Z}) \\ \mathcal{C}(\pi_1(\Sigma)) \arrow[rr, hook] & & \pi_1(\Sigma) \arrow[ur, "h"] \arrow[r, two heads, "\mathrm{proj}"'] & \pi_1(\Sigma)^{ab} \arrow[u, "\tilde{h}"', "\wr"] \end{tikzcd}$$

The composition in the bottom line is the zero map, which implies that $\Pi_*\colon H_1(\Sigma^{ab}, \mathbb{Z}) \to H_1(\Sigma, \mathbb{Z})$ is the zero map. Now we are done because the pull-back map $\Pi^*\colon H^1(\Sigma, \mathbb{R}) \to H^1(\Sigma^{ab}, \mathbb{R})$ is given by $(\Pi^*\alpha)(\sigma) = \alpha(\Pi_*\sigma)$, where $\sigma \in H_1(\Sigma^{ab}, \mathbb{Z})$, by the Universal Coefficients Theorem. □

We denote the *abelian Mañé critical value* by $c_{ab} = c(E^{ab})$. The abelian cover is particularly interesting in light of the following result:

Proposition 2.37. *It holds that $c_{ab} = c(E^{ab}) = c(L^{ab}) = c(E)$. In particular, σ is exact if and only if σ^{ab} admits a bounded primitive.*

Proof. If σ^{ab} does not admit a bounded primitive, then $c(L^{ab}) = c(E^{ab}) = c(E) = \infty$. Otherwise, $c(E^{ab}) < \infty$ and proposition 2.35 shows that σ is exact and, hence, the critical value $c(L)$ is defined (for any choice of Lagrangian). Let Θ be any closed 1-form on Σ. The above lemma states that the lift Θ^{ab} is exact. Thus, $c(L^{ab}) = c(L^{ab} - \Theta^{ab})$. Since Θ was arbitrary, remark 2.28, corollary 2.33, and proposition 2.35 imply

$$c(E) \overset{(2.35)}{=} c(E^{ab}) \overset{(2.33)}{\leq} c(L^{ab}) = \inf_{[\Theta]\in H^1(\Sigma,\mathbb{R})} c(L^{ab} - \Theta^{ab}) \overset{(2.28)}{\leq} \inf_{[\Theta]\in H^1(\Sigma,\mathbb{R})} c(L - \Theta) \overset{(2.33)}{=} c(E).$$

□

2.7 Distinguished Energy Levels

After investing a great deal of time in developing the theory surrounding Mañé's critical values, we finally harvest its payoff. We discussed earlier that for constant negative curvature and constant magnetic magnitude all but one energy level are HS-contact (proposition 2.3). In the general setting, we have the following result:

Proposition 2.38. *For any $k > c_{ab}$, the energy level (S_k, Ω) is HS-contact.*

Proof. By proposition 2.37, there is a primitive θ of σ with $\frac{1}{2}||\theta||_\infty^2 \leq k - \epsilon$ for some small $\epsilon > 0$. Then $\lambda = -\lambda_0 + \pi^*\theta$ is a primitive of the twisted symplectic form Ω. Since $\frac{1}{2}|v|^2 = k$ for vectors in S_k,

$$\begin{aligned}(\lambda_0 - \pi^*\theta)\,(F(x,v)) &= |v|^2 - \theta_x(v) \geq 2k - ||\theta||_\infty |v| \\ &\geq 2k - \sqrt{2(k-\epsilon)}\sqrt{2k} > 0\end{aligned}$$

and $\lambda(F)$ is never zero. Thus, $\iota_F(\lambda \wedge d\lambda) = \lambda(F)\Omega$ never vanishes and λ is a contact form. □

Therefore, for $k > c_{ab}$, the magnetic flow admits a contact time-change. It is noteworthy that this has to be a non-trivial (even non-canonical) time-change because the magnetic flow itself is never contact by theorem 2.18 (unless in the specific cases $s \equiv 0$ or s and K both constant). This is in contrast to corollary 2.24. Indeed, combining that corollary with the previous proposition yields:

Corollary 2.39. *Suppose the curvature is strictly negative but not constant and the magnetic magnitude is constant but not zero. Then $c_{ab} = \infty$.*

Proof. Since the geodesic flow is Anosov by the assumption on the curvature and since Anosov flows are structurally stable, for sufficiently large energy levels we enter the Anosov case. But now corollary 2.24 tells us that (S_k, Ω) is never HS-contact, so we cannot have a finite abelian Mañé critical value by the previous proposition. □

There may be a gap between c_{ab} and c_u. For energy values in this gap, being HS-contact is too strong of a property to ask for. Indeed, we have the following result ([Con06, Thrm. B.1]):

Proposition 2.40. *If Σ is not a torus and $c_u < k \leq c_{ab} < \infty$, then (S_k, Ω) is not HS-contact.*

Remark 2.41. *In [PP97, Section 4], the authors construct an example with an exact magnetic field on a genus two surface for which $c_u < c_a$ and for which some energy levels in between are Anosov. By the above proposition, the energy levels in between are not HS-contact. This provides an example as mentioned in remark 1.40.*

The remark shows that this gap is not pathological but may actually appear. Thus, for energy levels in between c_u and c_{ab} we need to study something weaker than HS-contact. To this end, we introduced the weaker notion of being virtually contact.

Proposition 2.42. *For any $k > c_u$, the energy level (S_k, Ω) is virtually contact.*

Proof. By hypothesis, there is some small $\epsilon > 0$ and a primitive θ of $\tilde{\sigma}$ with $\frac{1}{2}||\theta||_\infty^2 \leq k - \epsilon$. We will verify that $-\tilde{\lambda}_0 + \tilde{\pi}^*\theta$ is a suitable contact form on $\tilde{S}_k$. It certainly is bounded as both $\tilde{\lambda}_0$ and θ are. As in the last proposition, we conclude with the computation

$$\left(\tilde{\lambda}_0 - \tilde{\pi}^*\theta\right)(\tilde{F}(x,v)) \geq 2k - \sqrt{2(k-\epsilon)}\sqrt{2k} > 0.$$

□

We obtained information about when an energy level is virtually contact, HS-contact, or not HS-contact. However, we do not yet know anything specific about energy levels that are Anosov. It turns out that the Anosov property is linked to the virtually contact one.

Proposition 2.43. *If the magnetic flow on S_k is Anosov, then $k > c_u$. In particular, any Anosov energy level is automatically virtually contact.*

We deduce that remark 2.41 provides us with examples in which virtually contact is a strictly weaker notion than HS-contact. Further, this proposition gives us an upgrade of corollary 2.24, which yields examples of the same kind:

Corollary 2.44. *Suppose the curvature is strictly negative but not constant with supremum $K_{\max} < 0$ and the magnetic magnitude is constant but not zero. Then (S_k, Ω) is Anosov and virtually contact but not HS-contact.*

Before we can prove proposition 2.43, we need some auxiliary results. First, let us prove the following geometric lemma about the weak (un)stable manifolds of an Anosov magnetic flow.

Lemma 2.45. *If the magnetic flow on S_k is Anosov, then the weak (un)stable manifolds $W^{sc}(x,v)$ (respectively $W^{uc}(x,v)$) are Lagrangian submanifolds of $(T\Sigma, \omega)$.*

Proof. The proof is the same for the weak unstable as for the weak stable manifolds. Fix any point $(x,v) \in \Sigma$. There are some functions r_1, r_2 on Σ so that for any point $p \in W^{sc}(x,v)$

$$T_pW^{sc}(x,v) = E_p^c \oplus E_p^s = \operatorname{span}\langle F(p), r_1(p)H(p) + r_2(p)V(p)\rangle.$$

Now we just calculate

$$\omega(F, r_1H + r_2V) = -\langle F_V, r_1H_H\rangle + \langle Y(F_H), r_1H_H\rangle = 0$$

and conclude that $\omega(X,Y) = 0$ for any $X, Y \in T_pW^{sc}(x,v)$. This finishes the proof of the lemma. □

One further result we need is an observation due to Ehresmann about transverse foliations of fiber bundles inducing covering maps by restricting to leaves. For a proof, we refer to [CLN85, p. 91].

Proposition 2.46. *Suppose $p\colon E \to B$ is a fiber bundle with compact fiber F. Assume further that we are given a foliation of E of dimension $\dim(E) - \dim(F)$ that is transverse in the following sense: if L_x denotes the leaf passing through $x \in E$, then $T_xE = T_xF_x \oplus T_xL_x$. Then for any leaf L, the restriction $p|_L\colon L \to B$ is a covering map.*

Combining lemma 2.45 and proposition 2.46 allows us to prove proposition 2.43:

Proof of proposition 2.43. Consider the circle fibration $\pi\colon S_k \to \Sigma$. Since $T_{(x,v)}\pi^{-1}(x) = \text{span}(V(x,v))$ for any $(x,v) \in S_k$, the transversality result in theorem 2.10 implies that the foliation of weak stable manifolds W^{sc} of the magnetic flow is transverse to the above circle fibration in the sense that

$$T_{(x,v)}S_k = T_{(x,v)}\pi^{-1}(x) \oplus T_{(x,v)}W^{sc}(x,v)$$

at any point in S_k. Consider the universal cover $\tilde{\Sigma}$ and the corresponding $\tilde{S}_k \subset T\tilde{\Sigma}$. As the lift $\widetilde{W^{sc}}$ of the weak stable foliation is exactly the weak stable foliation of the lifted flow, the transversality continues to hold in the lift. By the result of Ehresmann, the restriction of $\tilde{\pi}$ to any leaf of $\widetilde{W^{sc}}$ is a covering map. Being simply connected, $\tilde{\Sigma}$ only admits trivial coverings. Therefore, $\tilde{\pi}|_{\widetilde{W^{sc}}(x,v)}\colon \widetilde{W^{sc}}(x,v) \to \tilde{\Sigma}$ is a diffeomorphism for any point $(x,v) \in \tilde{S}_k$. In particular, for any $(x,v) \in \tilde{S}_k$ and any $y \in \tilde{\Sigma}$ there is a unique point $(y, \mu_{x,v}(y))$ in $\widetilde{W^{sc}}(x,v) \cap \tilde{\pi}^{-1}(y)$. Now keep (x,v) fixed. Regard $y \mapsto \mu_{x,v}(y) \in T_y\tilde{\Sigma} \cong T_y^*\tilde{\Sigma}$ as a 1-form on $\tilde{\Sigma}$. Let us abbreviate $\mu = \mu_{x,v}$. By definition of μ, its graph in $T\tilde{\Sigma}$ is exactly $\widetilde{W^{sc}}(x,v)$. Moreover, because $\tilde{\pi}|_{\widetilde{W^{sc}}(x,v)}$ is a diffeomorphism, it follows that μ inherits the smoothness from $\widetilde{W^{sc}}(x,v)$ and that the submanifold structure, which $\widetilde{W^{sc}}(x,v)$ is given, coincides with the graph manifold structure from μ. We showed in lemma 2.45 that $\widetilde{W^{sc}}(x,v)$ is a Lagrangian submanifold of $(T\tilde{\Sigma}, \tilde{\omega})$. Recall that the Legendre transform $\tilde{\mathcal{L}}$ is a symplectomorphism $(T\tilde{\Sigma}, \tilde{\omega}) \to (T\tilde{\Sigma}, \tilde{\omega}_0)$ with $\tilde{\mathcal{L}}(\tilde{S}_k) = \tilde{H}^{-1}(k)$ (lemma 2.26). Thus, $\tilde{\mathcal{L}}(\widetilde{W^{sc}}(x,v)) \subset \tilde{H}^{-1}(k)$ is a Lagrangian submanifold of $(T\tilde{\Sigma}, \tilde{\omega}_0)$ and it is the graph manifold of the 1-form $\mu - \theta$. Since the graph of a 1-form is a Lagrangian submanifold of $(T\tilde{\Sigma}, \tilde{\omega}_0)$ if and only if the form is closed, $\mu - \theta$ must be closed. As $\tilde{\Sigma}$ has trivial first cohomology, $\mu - \theta$ even is exact. We have now shown that $\tilde{H}^{-1}(k)$ contains an exact Lagrangian graph, namely $\tilde{\mathcal{L}}(\widetilde{W^{sc}}(x,v))$. The symplectic characterization of the critical value, i.e. remark 2.31, yields $k \geq c_u$. Since Anosov flows are structurally stable, the magnetic flow on the energy level $S_{k'}$ is also Anosov for any k' sufficiently close to k. Therefore, we could have done the entire argument with k' instead of k to find $k > k' \geq c_u$. □

We found in remark 2.34 that the universal Mañé critical value is $\frac{-s^2}{2K}$ in the case of constant negative curvature and constant magnetic magnitude. Further, theorem 2.9 and proposition 2.40 imply that $c_{ab} = c_u$. The results about the various energy levels obtained in this section recover the observations made in theorem 2.9. However, we actually have not recovered all the information since we did not find out that any energy level above c_u is Anosov or that any energy level below c_u is HS-contact. To give an example that the less hands-on approach via Lagrangian dynamics is a strong tool nonetheless, let us finish the discussion about magnetic flows by verifying that we did not pass on any interesting Anosov flows by excluding the torus case in the first few chapters.

Corollary 2.47. *A torus does not admit an Anosov magnetic flow.*

Proof. By proposition 2.43, it suffices to show that $c_u = \infty$. The universal cover of a torus T is the euclidean plane $\Pi\colon \mathbb{R}^2 \to T$ and a Riemannian metric on T lifts to $k(x,y)||\cdot||_{\text{eucl}}$ for some bounded function k. A primitive of $\Pi^*(s\Omega_{\text{area}})$ is given by $f dy$, where $f(x,y) = \int_0^x (s \circ \Pi)(x', y) dx'$. By proposition 2.30, we obtain

$$c_u = \inf_{u \in C^\infty(\mathbb{R}^2, \mathbb{R})} \sup_{(x,y) \in \mathbb{R}^2} \frac{k(x,y)^2}{2} \left(||\partial_x u||^2_{\text{eucl}} + ||\partial_y u + f||^2_{\text{eucl}} \right).$$

This expression could only potentially be finite if $\partial_y u + f$ is bounded, but then u must be of the form $\int_0^y (g - f) dy'$ for some bounded function g. Then $\partial_x u$ is of the form $\int_0^y (\partial_x g) dy' + \int_0^y (s \circ \Pi) dy'$, where the first term is bounded, but the second term is unbounded in y, which in turn implies $c_u = \infty$. □

3 Anosov Dehn Surgery

In this chapter, we mean to inquire about the existence and construction of somewhat exotic Anosov flows in dimension three. In particular, manifolds are assumed to be 3-dimensional in the entire chapter. As discussed in the first chapter, the most prominent examples of flows are geodesic flows and suspensions. It is a remarkable result that, topologically, these actually constitute a very large collection of Anosov flows. Indeed, Ghys proved that any Anosov flow on a circle bundle has a finite cover that is orbit equivalent to a geodesic flow (theorem 2.11, [Ghy84, Thrm. A]). At the other end of the spectrum, Plante proved that any Anosov flow is orbit equivalent to a suspension if M has solvable fundamental group or if the bundle $E^s \oplus E^u$ is integrable (corollary 1.22, [Pla72, Thrm. 3.1], [Pla81, Thrm. B]). Seeing that a lot of Anosov flows are in nature a geodesic flow or a suspension, it is natural to ask whether this is the case for all Anosov flows. The answer is no and the first counter-example was constructed by Handel and Thurston ([HT80]). Quickly afterwards, several other authors published their construction of counter-examples including Goodman among others ([Goo83]). We can cast this question in a different light using the notion of algebraic flows (definitions will follow). Geodesic flows and suspensions both fit into this category, and Tomter proved that these are essentially the only ones (within the realm of Anosov flows) by showing that any algebraic Anosov flow has a finite cover that is orbit equivalent to one of these (thoerem 3.5, [Tom70]). Thus, the question becomes whether all Anosov flows are algebraic. By now there are quite a few counterexamples and one of the most prominent types of counter-examples are the ones living on graph-manifolds, meaning that they are obtained by a Dehn surgery. Handel and Thurston's as well as Goodman's example are of this type. We will now discuss them and rely on work of Foulon and Hasselblatt, who managed to encompass both counter-examples (and lots more) in a universal formulation ([FH13]).

3.1 The General Surgery

To conduct Dehn surgery, all we need is an annulus. However, we do not only want to obtain a new manifold but also a new Anosov flow. In order to do so, we will need an annulus A in M parametrized by two coordinates $s \in S^1$ and $w \in (-\epsilon, \epsilon)$ so that the vector fields $\frac{\partial}{\partial s}$ and $\frac{\partial}{\partial w}$ are E-transverse. We call a vector field *E-transverse* if it has non-zero stable and non-zero unstable component, i.e. if it is not strictly contained in either the weak stable or the weak unstable subbundle. If such an annulus is given, then we can apply the flow for small times to get a 3-dimensional open subset Λ parametrized by $r \in (-\eta, \eta)$, s, and w via $(r, s, w) = \phi_r((s, w))$. In this domain, the infinitesimal generator of ϕ_t is given by $\frac{\partial}{\partial r}$ by construction. Dehn surgery on the annulus A leaves the vector field $\frac{\partial}{\partial r}$ unchanged (as discussed in the appendix), so that F continues to define a vector field on the new manifold inducing a flow which agrees with ϕ_t outside the surgery domain. E-transversality will ensure that the new flow remains Anosov (see proposition 3.2 below).

In specific examples later on, we will usually construct these charts as follows. We call a knot *E-transverse* if its derivative is an E-transverse vector field. Now take any E-transverse knot γ in M (the existence of these is not obvious but given in most cases) and pick a local E-transverse vector field in a small neighborhood of the knot. Then this vector field induces a local flow ρ_t and we obtain a 2-dimensional neighborhood of the knot by $(s, w) = \rho_w(\gamma(s))$. This can either be a Möbius strip or an annulus. In the latter case, we have our desired chart. The parametrization $(r, s, w) \mapsto \phi_r \circ \rho_w \circ \gamma(s)$ is smooth because all involved maps are. Note that all admissible surgery domains can be build this way for $s \mapsto (0, s, 0)$ defines an E-transverse knot and $\frac{\partial}{\partial w}$ a local E-transverse vector field.

Remark 3.1. *Suppose the (un)stable subbundle is orientable. Given any E-transverse knot, we can then always ensure the existence of a surgery domain. Indeed, we can consider local strong (un)stable foliations along the knot and parametrize each by a w_s and a w_u coordinate, respectively. Then we build an E-transverse vector field on a 2-dimensional neighborhood of the knot by taking a non-trivial convex* *combination of $\frac{\partial}{\partial w_s}$ and $\frac{\partial}{\partial w_u}$. This can be turned into a local E-transverse vector field on a small 3-dimensional neighborhood simply by pushing it via $d\phi_t$ for small t. That the (un)stable subbundle is orientable ensures that the resulting chart is an annulus and not a Möbius strip.*

We claimed that E-transversality of $\frac{\partial}{\partial s}$ and $\frac{\partial}{\partial w}$ is sufficient to obtain a new Anosov flow. Let us first give a heuristic argument for the Anosov property of the new flow and provide a rigorous proof afterwards. Actually, the heuristic argument will not reveal why E-transversality is even needed and we need to wait for the rigorous argument to understand the reason. Morally, the exponential growth of the new flow is implied by the following three observations: firstly, outside the surgery domain, the new flow coincides with the old one and inherits the exponential growth from the latter; secondly, the surgery domain is a flow box and the flow will only stay inside for time 2η before it exits the surgery domain again; thirdly, because the flow is transverse to the annulus, it cannot immediately enter back into the surgery domain after it exited. In conclusion, as long as we flow outside the domain, we grow exponentially because the old flow does. When we enter, we may not grow at all, but at least we can bound the time in which we do not grow from above by the small constant 2η. After exiting the domain, we will stay outside for a certain amount of time that we can bound from below, by transversality and compactness. Whilst outside, we gain enough exponential growth to counter the fact we may enter back into the surgery domain. Thus, by adjusting the constant and the exponent in the growth estimate for the old flow, we can bound the plot of the growth of the new flow from below by another exponentially growing function. This is an illustration why we can expect the new flow to be Anosov. However, the Anosov property encompasses not only exponential growth but does this on an *invariant* splitting. To ensure the existence of an invariant splitting with the desired exponential growth, we need the E-transversality. The formal proof relies on the variant of the cone criterion using Lorentz forms (proposition 1.4).
Denote the shear map realizing the Dehn surgery by $D\colon A \to A$. Then D can be written as $D(s,w) = (s+T(w), w)$ for some smooth twist function $T\colon (-\epsilon,\epsilon) \to S^1$. We pose the (mild) assumption that T has strictly monotone derivative. Let $q \in \mathbb{Z}$ denote the surgery coefficient and $\psi_t\colon N \to N$ the flow obtained from the surgery on the new manifold N.

Proposition 3.2. *For any $q \leq 0$, the new flow ψ_t is Anosov.*

Proof. E-transversality implies that E^s and E^u have non-zero s and w direction when written as the span of a linear combination of $\frac{\partial}{\partial r}$, $\frac{\partial}{\partial s}$, $\frac{\partial}{\partial w}$. Thus, for a small choice of constant $c > 0$, the positive cone of the form $-cdr^2 \pm dwds$ contains E^s or E^u. By proposition 1.4, the original flow ϕ_t has corresponding quadratic Lorentz forms Q^s and Q^u. We may pick Q^s and Q^u so that their positive cones are also contained in the positive cone of $-cdr^2 \pm dwds$. Let us deform Q^s and Q^u by "widening" them close to the surgery domain so that they become $-cdr^2 \pm dwds$ inside the surgery domain. Then invariance of the cone field is still given since proposition 1.4 only requires it for times $t > t_0$ for some $t_0 > 0$. When we enter the surgery domain, we widen the positive cones and they remain trivially invariant. When we exit, the cones are not invariant for small times, but the Anosov property of ϕ_t ensures that the cones get contracted until they fit into the old cone field. The time after which this happens can be bounded from above, by compactness. Similarly, the growth conditions on ϕ_t in the cone fields are still satisfied since these are also only required for times $t > t_0$. This first argument is the crucial application of E-transversality.
Starting from these deformed Lorentz forms, we will construct new quadratic Lorentz forms on the new manifold obtained from the surgery. To this end, pick some smooth function $\rho\colon \mathbb{R} \to \mathbb{R}$ that is 1 on $(-\infty, 0]$, is 0 on $[\eta, \infty)$, and is strictly monotone decreasing in between 0 and η. Define

$$Q^{s,-} = -cdr^2 - dwds, \quad Q^{s,+} = -cdr^2 - \left(dwds + \rho(r)T'(w)dw^2\right),$$
$$Q^{u,-} = -cdr^2 + dwds, \quad Q^{u,+} = -cdr^2 + \left(dwds + \rho(r)T'(w)dw^2\right),$$

For $r < 0$ we can use $\rho \equiv 1$ to compute

$$(\mathrm{id} \times D)^* Q^{s,-} = -cdr^2 - dw\,(ds + T'(w)dw) = Q^{s,+}.$$

and, similarly, $(\mathrm{id} \times D)^* Q^{u,-} = Q^{u,+}$. Moreover, $Q^{s,-}$ and $Q^{u,-}$ connect at $r = -\eta$ neatly with Q^s and Q^u, as well as $Q^{s,+}$ and $Q^{u,+}$ do at $r = \eta$ since ρ is zero on $[\eta, \infty)$. Thus, the form P^s defined by being Q^s outside the surgery chart, $Q^{s,-}$ on $\{r < 0\}$, and $Q^{s,+}$ on $\{r > 0\}$ is a well-defined quadratic form on the new manifold N as is the analogously defined P^u. We will now verify the hypothesis of proposition 1.4 for these quadratic forms and the new flow ψ_t. Of the four assumptions, the second and third are immediate. Let us check the fourth. As usual, we restrict our attention to the stable case and the other case is dealt with analogously. The use of a flow box ensures that $d\psi_t$ is just the identity. Suppose we are given a non-zero vector $v \in \overline{C_x^{s,+}}$, where $x = (r, s, w) \in \Lambda$, and write it as $v = v_r \frac{\partial}{\partial r} + v_s \frac{\partial}{\partial s} + v_w \frac{\partial}{\partial w}$. This is equivalent to saying

$$0 \leq Q_x^{s,+}(v) = -c v_r^2 - \left(v_w v_s + \rho(r) T'(w) v_w^2 \right).$$

In particular, $v_w \neq 0$. Note that by the assumption that T' is strictly monotone, the hypothesis $q \leq 0$ is equivalent to saying $T' < 0$. Then, as long as $\psi_t(x)$ remains in in the right side of the surgery chart,

$$Q^{s,+}_{\psi_{-t}(x)}\left((d\psi_{-t})_x(v)\right) = -c v_r^2 - \left(v_w v_s + \rho(r-t) T'(w) v_w^2 \right) \overset{T'<0}{\geq} \underbrace{\left(\rho(r-t) - \rho(r)\right)}_{>0} |T'(w)| v_w^2 > 0,$$

which shows $(d\psi_{-t})_x \left(\overline{C_x^{s,+}} \setminus \{0\} \right) \subset C^{s,+}_{\psi_{-t}(x)}$ for all times t with $\psi_{-t}(x) \in \{r > 0\}$. On the other side of the surgery,

$$Q_x^{s,-}(v) = -c v_r^2 - v_w v_s = Q^{s,-}_{\psi_{-t}(x)}\left((d\psi_{-t})_x(v)\right),$$

i.e. $d\psi_{-t}$ leaves the cones exactly invariant but does not contract them. This issue is again countered by only requiring invariance for $t > t_0$ and knowing that we have strict invariance inherited from ϕ_t outside the surgery domain. This establishes assumption number four in proposition 1.4. The same argument asserts the first hypothesis of the proposition. We already calculated that $d\psi_{-t}$ is non-contracting on the stable cone. Together with the discussion of the heuristic argument for exponential growth of ψ_t, we conclude exponential growth in the cone field for times $t > t_0$. $\square$

Remark 3.3. *Note that the need for $q \leq 0$ is a technical issue ensuring that we twist in the correct direction in order not to counter the evolution of the Lorentz forms.*

Remark 3.4. *Since we can take the Lorentz forms for ϕ_t arbitrarily small outside the surgery domain, we find that the Anosov splittings of ψ_t and ϕ_t coincide on $M \setminus \Lambda = N \setminus \Lambda$.*

Example: Handel and Thurston

Handel and Thurston performed Dehn surgery on a geodesic flow living on the unit tangent bundle $U\Sigma$ of a closed oriented surface (Σ, g) of constant negative curvature ([HT80]). Let J denote the associated canonical almost complex structure. Given any simple closed geodesic c, consider the knot in $U\Sigma$ given by $\gamma(s) = (c(s), l_0 J\dot{c}(s))$, where $l_0 > 0$ is taken so that $l_0 |\dot{c}| \equiv 1$. This is an E-transverse knot by theorem 2.6. The same theorem shows that the vertical vector field $V(x, u) = (0, Ju)$ is E-transverse. If ρ_t denotes its flow, then we get a 2-dimensional neighborhood by $(s, \nu) = \rho_\nu(\gamma(s)) = (c(s), l_0 e^{i\nu} J\dot{c}(s))$. Later on, it will come in handy to perform a change of variables by $w = \frac{1}{l_0} \cos(\nu + \pi/2)$. Then the 3-dimensional neighborhood is given by $\phi_r(c(s), l_0 e^{i(\nu(w) + \pi/2)} \dot{c}(s))$. Note that using the exponential map we can write down an explicit chart map by

$$\Lambda \to U\Sigma, \ (r, s, w) \mapsto \left(\exp_{c(s)} \left(r l_0 e^{i(\nu(w) + \pi/2)} \dot{c}(s) \right), \frac{\partial}{\partial r} \exp_{c(s)} \left(r l_0 e^{i(\nu(w) + \pi/2)} \dot{c}(s) \right) \right).$$

Actually, Handel and Thurston had no need for such an explicit description of their surgery chart, but we will make use of it when we later study this example in light of Foulon and Hasselblatt's surgery. In that same instance we will also verify that the new flow ψ_t is volume-preserving. The former authors used this surgery to produce the first non-algebraic Anosov flows. This requires picking a simple closed geodesic c that is also separating. We will reproduce their argument that the resulting flow is no longer algebraic further below.

Example: Goodman

Goodman also produced new flows by using Dehn surgery, but her construction works more generally for any Anosov flow that admits a periodic orbit, not just for geodesic flows ([Goo83]). She constructs her surgery domain as follows. Start with any periodic orbit of ϕ_t. Consider the toral boundary of a small tubular neighborhood of this orbit. Given any point on the orbit, imagine plotting the unstable subbundle on the x-axis, the stable subbundle on the y-axis, and the the flow direction running in z-direction through the sheet. Then the flow lines starting north of the orbit will traverse in a hyperbolic manner towards the y-axis in the south while expanding to the east along the y-axis (i.e. qualitatively behave like the graph of $1/x$). By deforming the torus in the upper right quadrant so that a small part of its boundary looks like the graph of the identity function, it will become transverse to the flow in this small part (see the pictures in [Goo83, p. 301] and [Bar17, p. 7]). We can parametrize this part by $(-\epsilon, \epsilon)$ and since we do this at any point of the orbit we obtain an annulus $S^1 \times (-\epsilon, \epsilon)$. Because we performed the deformation in the upper right quadrant, the two vector fields $\frac{\partial}{\partial s}$ and $\frac{\partial}{\partial w}$ are E-transverse. Goodman then conducts Dehn surgery on this annulus.

3.2 Non-Algebraic Flows

In this section, we will reproduce the proof by Handel and Thurston that their new Anosov flows are not algebraic. But first we need to introduce algebraic flows to begin with. We say that a flow on a closed manifold is *algebraic* if the manifold can be written as $\Gamma \setminus G/K$, where G is a Lie group, $K < G$ a compact subgroup, and $\Gamma < G$ a discrete cocompact subgroup acting by left-multiplication, and if the flow is given by $\Gamma g K \mapsto \Gamma g \exp(t\alpha) K$ for some α in the Lie algebra associated to G. This class of flows contains both geodesic flows and suspensions. The result by Tomter we announced in the introduction is the following, see [Tom70, Thrm. IV+V] (this is not restricted to dimension three).

Theorem 3.5. *Suppose we are given an algebraic Anosov flow. If the associated Lie algebra is semi-simple, then the lift of the algebraic flow to some finite cover is C^1-conjugate to the geodesic flow on the unit tangent bundle of some closed manifold. On the other hand, if the associated Lie algebra is solvable, then the lift of the algebraic flow to some finite cover is, up to a constant time-change, C^1-conjugate to the suspension flow of a bundle over the circle with fibers an infra-nilmanifold.*

This theorem emphasizes that the existence of an algebraic Anosov flow is a serious constraint on the underlying manifold. We will explore this knowledge to prove that our new flow cannot be algebraic by investigating the new manifold N. Before that, let us extract a more concise statement suitable for our purposes from the above theorem. We call a 3-manifold *Seifert-fibered* if it can be decomposed into a disjoint union of circles such that for each fiber there is a tubular neighborhood diffeomorphic to the torus obtained from $D^2 \times [0, 1]$ by identifying $D^2 \times \{0\}$ and $D^2 \times \{1\}$ via a rational rotation; moreover, we require the diffeomorphism to be fiber-preserving in the sense that $D^2 \times \{s\}$ gets mapped to one of the circles in the disjoint union. Thus, the above theorem translates to this corollary.

Corollary 3.6. *Suppose we are given an Anosov flow in dimension three. If no finite cover of the underlying manifold is homeomorphic to a Seifert-fibered manifold or a torus bundle over the circle, then the flow is not algebraic.*

Proof. Since we are in dimension three, any real Lie algebra is either semi-simple or sovlable[4] and, hence, we can apply theorem 3.5. Either we have a lift conjugate to a geodesic flow, in which case we end up with a Seifert-fibered manifold, or we have a lift conjugate to a suspension flow, in which case we end up with a torus bundle over the circle by proposition 1.14. □

Note that the first part of the corollary is essentially a negative analogue to Ghys' theorem 2.11, whereas the second part of the corollary together with the next lemma poses a negative analogue to Plante's theorem, [Pla81, Thrm. B]. We will use some tools from algebraic topology to reformulate the conditions in the corollary to conditions on the fundamental group of the manifold.
First, we recall an essential ingredient from covering space theory. Suppose $p\colon \hat{M} \to M$ is a covering space of a manifold M. Then we can consider the fundamental group of $\hat{M}$ as a subgroup of the fundamental group of M via the injective map induced by p. It follows from the classification of covering spaces that $p\colon \hat{M} \to M$ is a finite cover if and only if $\pi_1(\hat{M}, \star)$ is a subgroup of $\pi_1(M, \star)$ of finite index.

Lemma 3.7. *If M is a 3-manifold whose fundamental group contains no solvable finite index subgroup, then no finite cover of M is homeomorphic to a torus bundle over the circle.*

Proof. Suppose $\hat{M}$ is a torus bundle over the circle. Then the long exact sequence in homotopy (neglecting base-points)

$$\cdots \to \pi_2(S^1) \to \pi_1(T^2) \to \pi_1(\hat{M}) \to \pi_1(S^1) \to \pi_0(T^2) \to \cdots$$

becomes

$$0 \to \mathbb{Z} \oplus \mathbb{Z} \hookrightarrow \pi_1(\hat{M}) \to \mathbb{Z} \to 0.$$

From this, we obtain an isomorphism $\pi_1(\hat{M})/(\mathbb{Z}\oplus\mathbb{Z}) \to \mathbb{Z}$. Since the factor group is abelian, it is solvable, as is $\mathbb{Z}$. Then the fundamental group $\pi_1(\hat{M})$ itself must be solvable. □

We can upgrade this to a statement about the existence of free subgroups of the fundamental group. Here is the algebraic lemma enabling this:

Lemma 3.8. *If H is a finite index subgroup of a group G and if F is a free subgroup of G, then H contains a subgroup of F that is isomorphic to F.*

Proof. This is basically obvious. Suppose that F is generated by the elements $\{g_\alpha\}_\alpha \subset G$. If for some α we had $g_\alpha^n \notin H$ for all $n \in \mathbb{Z}$, then $g_\alpha^n H$ and $g_\alpha^k H$ are distinct elements of G/H for $n \neq k$. But this contradicts the assumption that the index of H in G is finite. Hence, for all α there is some $n_\alpha \in \mathbb{Z}$ with $g_\alpha^{n_\alpha} \in H$, and the elements $\{g_\alpha^{n_\alpha}\}_\alpha$ generate a subgroup of H isomorphic to F. □

Lemma 3.9. *If M is a 3-manifold whose fundamental group contains a free non-abelian group, then no finite cover of M is homeomorphic to a torus bundle over the circle.*

Proof. By lemma 3.7, we need to show that the existence of a free non-abelian subgroup implies that there cannot be any solvable finite index subgroup. Any group can be written as the quotient of a free group by a normal subgroup. Since the alternating group A_5 is the quotient of a free group with two generators, any free non-abelian group has a quotient that is isomorphic to A_5. In particular, as A_5 is not solvable, any free non-abelian group cannot be solvable. We can conclude with the previous algebraic lemma. □

We would also like to obtain conditions on the fundamental group that replace the hypothesis of having no finite cover homeomorphic to a Seifert-fibered manifold. This is done as follows:

[4] For instance, this is a consequence of theorem 1.42 and proposition 1.43 in [Kna96]. Being in dimension three, the hypothesis of the aforementioned proposition 1.43 are equivalent to having a degenerate Killing form.

Lemma 3.10. *Suppose that M is a 3-manifold whose fundamental group is infinite. If $\pi_1(M, \star)$ has no finite index subgroup that itself contains an infinite cyclic normal subgroup, then no finite cover of M is homeomorphic to a Seifert-fibered manifold.*

Proof. A result in [Sco83, Lem. 3.1] states that if $\hat{M}$ is a Seifert-fibered manifold, then its universal cover is homeomorphic to S^3, $\mathbb{R}^3$, or $S^2 \times \mathbb{R}$. If the universal cover is the 3-sphere and, in particular, is compact, then the fundamental group of $\hat{M}$ is finite. In the other two cases, $\pi_1(\hat{M}, \star)$ contains an infinite cyclic normal subgroup as shown in [Sco83, Lem. 3.2]. □

In conclusion, we can rewrite corollary 3.6 as:

Corollary 3.11. *Suppose the fundamental group of M contains a free non-abelian subgroup but does not contain a finite index subgroup that itself contains an infinite cyclic normal subgroup. Then M does not admit any algebraic Anosov flows.*

Having set up the algebraic tools, we are ready to tackle our new manifold N.

Theorem 3.12. *Suppose M is the unit tangent bundle $U\Sigma$ of a negatively curved, oriented, closed, surface Σ. If the knot along which we perform surgery projects to a separating curve, then the new flow obtained from the surgery is not algebraic.*

Proof. By corollary 3.6, we need to rule out that N has a finite cover homeomorphic to a Seifert-fibered manifold or a torus bundle over a circle. That (a cover of) N does not magically become a torus bundle over a circle is rather intuitive. Indeed, ruling out this case by finding a free non-abelian subgroup of its fundamental group (see lemma 3.9) is rather straightforward. The more difficult part is to show that the surgery breaks the circle bundle structure.
We want to study the fundamental group of the new manifold N in order to apply corollary 3.11. Since the projected knot is separating, it divides the surface into two components Σ_1 and Σ_2. Each Σ_i is a punctured surface and, hence, the fundamental group of Σ_i is some free non-abelian group F_i. Let M_1 and M_2 denote the unit tangent bundle of each component. Inspecting the long exact sequence in homotopy (neglecting base-points) yields

$$\cdots \to \pi_2(\Sigma) \to \pi_1(S^1) \to \pi_1(M_i) \to \pi_1(\Sigma_i) \to \pi_0(S^1) \to \cdots .$$

The group $\pi_2(\Sigma)$ is zero because the long exact sequence for the universal cover $\mathbb{H} \to \Sigma$ is

$$\cdots \to \pi_n(\mathbb{H}) \to \pi_n(\Sigma) \to \pi_{n-1}(F_{\text{dis}}) \to \cdots ,$$

where F_{dis} is the corresponding discrete fiber, and the groups $\pi_n(\mathbb{H})$ and $\pi_{n-1}(F_{\text{dis}})$ are both zero for $n \geq 2$. Thus, the first sequence reads

$$0 \to \mathbb{Z} \hookrightarrow \pi_1(M_i) \to F_i \to 0.$$

Since we are dealing with the unit tangent bundle of an oriented surface, given a loop in Σ_i, we can parametrize the fibers along the entire loop. In other words, we can parametrize the restriction of M_i to the loop by two S^1-coordinates, one for the loop and one for the fibers. This shows that the above short exact sequence splits at the first two stages and, consequently, $\pi_1(M_i) \cong F_i \oplus \mathbb{Z}$. Let M_0 denote the common boundary of M_1 and M_2. By taking each M_i to be just a slightly larger open set, van Kampen's theorem implies $\pi_1(M) \cong \pi_1(M_1) *_{\pi_1(M_0)} \pi_1(M_2)$. What does this product with amalgamation look like? Let r denote the generator of the $\mathbb{Z}$-component in $\pi_1(M_1)$ and likewise r' for $\pi_1(M_2)$. M_0 is a torus with one component corresponding to the separating knot and one to the fibers. The generator of the second one simply gets mapped to r in $\pi_1(M_1)$. On the other hand, by construction of the surgery, its image in $\pi_1(M_2)$ is sr', where s denotes the word in F_2 that corresponds to the image of the knot under the restricted shear map $D|_{\{s_0\}\times(-\epsilon,\epsilon)}$. Put differently, s corresponds to the word that represents the curve winding q times around the puncture in Σ_2. Thus, as elements of $\pi_1(M)$ we have $r = sr'$. Let w_0 denote the word in F_1 that represents $\partial\Sigma_1$, i.e. the projection of the knot along which we performed surgery. There is a corresponding word w_0' in F_2 for $\partial\Sigma_2$. The generator of the knot component in $\pi_1(M_0)$ gets mapped to w_0 in F_1 and to w_0' in F_2. Thus, we see that

$$\pi_1(M) \cong ((F_1 \oplus \mathbb{Z}) * (F_2 \oplus \mathbb{Z})) / \langle r = sr',\ w_0 = w_0'\rangle.$$

In particular, $\pi_1(M)$ is still large enough in the sense that it contains a free non-abelian subgroup. It remains to show that any finite index subgroup $H < \pi_1(M)$ cannot contain an infinite cyclic normal subgroup. Suppose for contradiction there was such $\langle g\rangle < H$. A priori, g is some word of the form $h_1h_1' \dots h_nh_n'$ with $h_j = \omega_j r^{i_j} \in F_1 \oplus \mathbb{Z}$ and $h_j' = \omega_j'(r')^{i_j'} \in F_2 \oplus \mathbb{Z}$. However, we can rewrite any expression of the form $r\omega'$ as $sr'\omega' = s\omega' r' = s\omega' s^{-1} r$. Note that $s\omega' s^{-1}$ is again a word in F_2 so that any combination of the form $\omega r^k \omega'(r')^m$ is the same as $\omega\tilde{\omega}' r^k (r')^m$ for some new word $\tilde{\omega}'$ in F_2. Moreover, rr' is the same as $rs^{-1}r$, which can again be converted to a word $\tilde{s}r^2$ for some word $\tilde{s}$ in F_2. Hence, this shows that $\omega r^k \omega'(r')^m$ is equivalent to a word of the form $\omega\tilde{\omega}' r^{k+m}$. In particular, we can convert g into an element wr^k, where w is a word of generators in F_1 and F_2 only. Since $\langle g\rangle$ is assumed to be normal, for any $h \in H$ there is some integer p_h such that $hgh^{-1} = g^{p_h}$. Then also $g^{p_{h_1h_2}} = g^{p_{h_1}} g^{p_{h_2}}$, i.e. $p_{h_1h_2} = p_{h_1}p_{h_2}$ for any $h_1, h_2 \in H$. In particular, any p_h can only be $+1$ or -1. After passing to the subgroup $\{h \in H \mid p_h = 1\}$ of H, we still work in a finite index subgroup of $\pi_1(M)$. Therefore, we may assume that p_h is always 1. By lemma 3.8, $F_1 \cap H$ contains a free non-abelian group F_H. Since F_1 and r commute, we have for any $h \in F_H$ the equalities $whr^k = wr^kh = gh = hg = hwr^k$. In particular, $wh = hw$ for all $h \in F_H$. Since w is a word of generators in F_1 and F_2, this can only hold if $w = \mathrm{id}$ or if h is a power of w. However, since F_H is free non-abelian, it cannot only contain powers of w. We conclude that $w = \mathrm{id}$ and so $g = r^k$. The entire above argument is symmetric in $i = 1, 2$, so we also obtain an integer m with $g = (r')^m$. But the equality $(r')^m = g = r^k = (sr')^k = s^k(r')^k$ can only be true if s is the identity. This, in turn, implies that the surgery was topologically trivial, but it was a Dehn surgery with coefficient $q \neq 0$. This leads to a contradiction, proving that there is no finite index subgroup in $\pi_1(M)$ that contains an infinite cyclic normal subgroup. $\square$

Remark 3.13. *We did not just prove that the new flow is not algebraic, but also that it is not orbit equivalent to an algebraic flow. For convenience, let us include this in the definition and from now on say that a flow is algebraic if it is orbit equivalent to an algebraic flow in the previous sense. The conclusion of the theorem still holds with this new definition.*

3.3 Surgery for Stable Hamiltonian Structures

We have seen how to conduct Dehn surgery preserving the Anosov property of a given flow and even giving rise to non-algebraic flows. In this chapter, we want to specialize the surgery so that it preserves more geometric structure of the manifold. More precisely, we are interested in Hamiltonian structures. Suppose ϕ_t is a Reeb flow of an Anosov Hamiltonian structure (M, Ω). Let a surgery domain Λ for Anosov Dehn surgery parametrized by $r \in (-\eta, \eta)$, $S \in S^1$, and $\nu \in (-\epsilon', \epsilon')$ be given. Since $F = \frac{\partial}{\partial r}$ spans the kernel of Ω, there is a smooth non-vanishing function g on Λ with $\Omega = g d\nu \wedge dS$ in Λ. The function g does not depend on r because Ω is closed. Introduce $w(S,\nu) = \int_0^\nu g(S,\nu')d\nu'$. Then $w(S, 0) \equiv 0$ and $\frac{\partial}{\partial\nu}w = g$, which never vanishes, so $(r, S, \nu) \mapsto (r, S, w(S, \nu)) = (r, s, w)$ is a well-defined coordinate transformation. If we denote the interval parametrizing w by $(-\epsilon, \epsilon)$, then the notation matches up with the previous one. The vector field generated by the w-coordinate is $\frac{1}{\partial_\nu w}\frac{\partial}{\partial\nu}$, which is still E-transverse. In the new coordinate system, the vector field generated by the s-coordinate becomes $\frac{\partial}{\partial S} - \frac{\partial_S w}{\partial_\nu w}\frac{\partial}{\partial\nu}$. Thus, at coordinates with $\nu = 0$ we have $\frac{\partial}{\partial s} = \frac{\partial}{\partial S}$ and by shrinking ϵ we can ensure that $\frac{\partial}{\partial s}$ stays E-transverse. In conclusion, up to shrinking the width, we have the same surgery domain as before and it is still suitable for Anosov Dehn surgery. In the new coordinate system, $\Omega = dw \wedge ds$. Since

$$(\mathrm{id} \times D)^*(dw \wedge ds) = dw \wedge d(s + T(w)) = dw \wedge ds,$$

the form Ω is preserved by the surgery. Therefore, after conducting the Anosov Dehn surgery, we get a new Hamiltonian structure (N, Ω). This Hamiltonian structure is Anosov because the new flow ψ_t is one of its Reeb flows. In other words:

Proposition 3.14. *Up to shrinking and reparametrizing the surgery annulus, Anosov Dehn surgery preserves Anosov Hamiltonian structures.*

We presented a surgery adapted to a Hamiltonian structure. A potential issue that comes to mind is that we fixed an arbitrary Reeb flow but the surgery should in some sense be independent of the choice of Reeb flow so that it really is subject to the Hamiltonian structure only. When we proved in proposition 1.1 that the Anosov property persists under time-changes, the proof revealed that the (un)stable subbundle only differs in a contribution in the flow direction. In particular, if a vector field is E-transverse for a given Anosov flow, then it is also E-transverse for any time-change of that flow. Suppose ϕ'_t is another Reeb flow. By the preceding argument, the annulus remains suitable for Anosov Dehn surgery on ϕ'_t. However, the r-coordinate can cause trouble. The infinitesimal generator of ϕ'_t in the above coordinates is $R(r, s, w)\frac{\partial}{\partial r}$ for some smooth function R that may depend on all coordinates. In particular, this vector field is not preserved by the surgery if it has a non-trivial s-dependence. As observed in the appendix, we notice that doing surgery in a ϕ'_t-flow box around A we do not even end up with the same manifold N. Fortunately, the proceeding discussion in the appendix makes sure the surgery is independent of the choice of Reeb flow up to smooth conjugacies:

Lemma 3.15. *A flow ψ'_t coming from ϕ'_t-surgery, where ϕ'_t is a time-change of ϕ_t, is smoothly conjugate to a time-change of ψ_t. Conversely, for any time-change ψ'_t of ψ_t, there is a time-change ϕ'_t of ϕ_t so that ψ'_t is smoothly conjugate to a constant time-change of the flow coming from ϕ'_t-surgery.*

Proof. As noted in the appendix, doing ϕ'_t-surgery gives rise to a different manifold N', but any diffeomorphism h_{loc} of the respective flow boxes with a restriction only at the boundary gives rise to a global diffeomorphism $N \to N'$ that carries the infinitesimal generator of ψ_t to a multiple of the infinitesimal generator of ψ'_t. On the other hand, given a time-change ψ'_t of ψ_t, we can take a time-change ϕ'_t of ϕ_t so that the infinitesimal generators of ψ'_t and ϕ'_t agree on $N \setminus \Lambda = M \setminus \Lambda$. As before, we perform ϕ'_t-surgery to obtain N' and take a local diffeomorphism of the flow boxes (this time from Λ' to Λ). As discussed in the appendix, if the diffeomorphism is of the form $h_{\mathrm{loc}}(r', s, w) = (h_1(r', s, w), s, w)$, then the global diffeomorphism $N' \to N$ maps $\frac{\partial}{\partial r'}$ to $\left(\frac{\partial}{\partial r'} h_1\right) \frac{\partial}{\partial r}$. The infinitesimal generators already agree at the boundary of Λ' and since we have no restriction on h_1 in the interior, we can take it so that it maps $\frac{\partial}{\partial r'}$ to the infinitesimal generator of ψ'_t. Strictly speaking, we cannot take h_1 arbitrarily inside Λ' since it still needs to be a well-defined function taking values in Λ. However, since we are free to take ϕ'_t as we want inside Λ' and may replace it by a constant time-change, we may take it so that the needed h_1 can be realized. $\square$

Next, we would like to preserve more than the just the Hamiltonian structure, namely stability. By theorem 1.36, this reduces to studying the HS-contact and the suspension cases. We begin with the former.

Example: Foulon and Hasselblatt

In this section, we discuss the surgery designed by Foulon and Hasselblatt. They were concerned with adapting the surgery so that it preserves the contact property of a flow. Suppose ϕ_t is the Reeb flow of a contact form λ on M and, further, that the (un)stable subbundle is orientable. Surgery domains suitable for this type are given by those whose E-transverse knot[5] is *Legendrian*, which means that its derivative lies in the contact structure, and for which the vector field $\frac{\partial}{\partial \nu}$ is contained in $E^s \oplus E^u$ at coordinates $r = 0$. Remark 3.1 shows that such surgery domains always exist once we have an E-transverse Legendrian knot. Given such a surgery domain, we can adapt it to the Anosov Hamiltonian structure $d\lambda$ as above.

Lemma 3.16. *After adapting the surgery domain to the Anosov Hamiltonian structure $d\lambda$,*

$$\lambda = dr + w ds, \quad d\lambda = dw \wedge ds, \quad \lambda \wedge d\lambda = dr \wedge dw \wedge ds.$$

Proof. Consider (r, S, ν)-coordinates constructed as described in remark 3.1 before we reparametrize the surgery domain to adapt it to the Anosov Hamiltonian structure $d\lambda$. In these coordinates, we obtain some local expression

$$\lambda = f_0(r, S, \nu)dr + f_1(r, S, \nu)dS + f_2(r, S, \nu)d\nu.$$

Since $F = \frac{\partial}{\partial r}$ and $\lambda(F) \equiv 1$, we must have $f_0 \equiv 1$. The vector field $\frac{\partial}{\partial \nu}$ lies in $E^s \oplus E^u$ at coordinates $r = 0$ by assumption, which implies $f_2(0, S, \nu) \equiv 0$. Since F is the Reeb vector field of λ, it holds that $\iota_F d\lambda = 0$. This translates to f_1 and f_2 being independent of r. Therefore, $f_2 \equiv 0$ and λ takes the expression $dr + f_1(S, \nu)dS$. In particular, $d\lambda = (\partial_\nu f_1)d\nu \wedge ds$. Adapting the surgery domain to the Anosov Hamiltonian structure $d\lambda$ is, hence, done by $w(S, \nu) = f_1(S, \nu) - f_1(S, 0)$. But $f_1(S, 0) = \lambda\big(\frac{\partial}{\partial S}\big)|_{(0,S,0)}$ is zero because the knot for the surgery is Legendrian and $\frac{\partial}{\partial S}|_{(0,S,0)}$ is exactly its derivative. Thus, $\lambda = dr + w ds$ in the adapted coordinates. □

We compute

$$(\mathrm{id} \times D)^* \lambda = dr + w d(s + T(w)) = \lambda + w T'(w) dw$$

to find that the contact form gets broken by the surgery. However, we can also observe that the volume form $\lambda \wedge d\lambda$ is clearly preserved by the surgery, so the new flow ψ_t is always volume preserving when we start with a contact flow. Indeed, this follows readily from Cartan's formula. Still, we hope to recover the contact property. The form λ may get broken but we know how it breaks. The idea now is as follows: we perturb λ on one side of the surgery domain $\{r < 0\}$ to λ_- and on the other side $\{r > 0\}$ to λ_+. The resulting object no longer defines a 1-form on M due to the discontinuity at $r = 0$, but if we manage to ensure $(\mathrm{id} \times D)^* \lambda_- = \lambda_+$, then this object will glue to define a 1-form after the surgery. Since the Hamiltonian structure $d\lambda$ is preserved by the surgery, we want to perturb λ so that the resulting form on the new manifold still is a primitive of $d\lambda$. For instance, this can be achieved by perturbing λ by an exact form on each side. Introduce the correction function

$$c(r, w) = \frac{1}{2}\rho(r) \int_{-\epsilon}^{w} x T'(x) dx,$$

where $\rho\colon \mathbb{R} \to \mathbb{R}$ is a smooth bump function that is 1 close to the origin and whose support is contained in $(-\eta, \eta)$. We pose two requirements on the twist map T: firstly, its derivative T' should be symmetric about the origin; secondly, T' should smoothly tend to 0 as $w \to \pm\epsilon$. Then $c(r, w) \to 0$ smoothly as $r \to \pm\eta$ or as $w \to \pm\epsilon$. Hence, we can extend c by zero to a smooth function defined on the entire manifold M. Close to $r = 0$, where ρ is constantly 1, we have $dc = \frac{1}{2} w T'(w) dw$, i.e. the term by which the contact form gets broken under the surgery, up to a factor $\frac{1}{2}$. Since c has no s-dependence, it is preserved by the surgery, as is dc. If we take $\lambda \pm dc$ as our perturbations $\lambda_\pm$, then indeed

$$(\mathrm{id} \times D)^*(\lambda_-) = \lambda + w T'(w) - dc = \lambda_+.$$

[5] It is noteworthy that any knot is isotopic to a Legendrian knot, though not necessarily to an E-transverse one ([Gei08, p. 101]).

Moreover, as noted above, the new 1-form on the new manifold N is a primitive of $dw \wedge ds = d\lambda$. What is not clear is that the new form is a contact form. For this to be true, we need that

$$(\lambda \pm dc) \wedge d(\lambda \pm dc) = \lambda \wedge d\lambda \pm \left(\frac{\partial}{\partial r} c\right) dr \wedge d\lambda = \left(1 \pm \frac{\partial}{\partial r} c\right) dr \wedge dw \wedge ds$$

never vanishes. Let us show that this can be achieved with a particular choice of twist map and by shrinking the width of the surgery annulus.

Lemma 3.17. *When choosing ϵ, T, and ρ suitably, $\frac{\partial}{\partial r} c$ has absolute value strictly less than 1.*

Proof. Let $|\rho'|_\infty$ and $|T'|_\infty$ denote uniform upper bounds for the derivatives of ρ and T, respectively. Then

$$\left|\frac{\partial}{\partial r} c\right| = \left|\frac{1}{2}\rho'(r) \int_{-\epsilon}^{w} xT'(x)dx\right| \leq |\rho'|_\infty |T'|_\infty \int_0^\epsilon x dx = \frac{1}{2}\epsilon^2 |\rho'|_\infty |T'|_\infty.$$

Clearly, if ϵ is small enough, then this expression is smaller than 1. Though, we do need to pay attention to the choices of ρ and T because ϵ is the first variable that we fixed and needs to be picked independently of later choices. The bump function can be taken so that the bound $|\rho'|_\infty$ depends only on η and is, say, smaller than $2/\eta$. We construct T as follows. Take a smooth function $t\colon \mathbb{R} \to [0, 2\pi]$ which is 0 on $\{x \leq -1\}$ and is 2π on $\{x \geq 1\}$. The uniform bound $|t'|_\infty$ of t' is independent of any other choice. We may take it to be smaller than 4. Moreover, pick t so that t' is symmetric around the origin. Set, for instance, $T(w) = \exp(-iqt(\frac{w}{\epsilon}))$, where q is the surgery coefficient. Then T satisfies the requirement of having symmetric derivative about the origin and that $T(w) \to 0$ and $T'(w) \to 0$ smoothly as $w \to \pm\epsilon$. Moreover, $|T'|$ is uniformly bounded by $\frac{q}{\epsilon}|t'|_\infty$. Thus,

$$\left|\frac{\partial}{\partial r} c\right| \leq \frac{1}{2}\epsilon^2 |\rho'|_\infty |T'|_\infty \leq \frac{4q}{\eta}\epsilon.$$

We see that, in the beginning of the surgery process, we could have first fixed a small η, and then shrunken ϵ to be smaller than $\frac{\eta}{4q}$. After these choices have been made, we can pick ρ and T as above so that $\frac{\partial}{\partial r} c$ has absolute value strictly less than 1. $\square$

We finished constructing a contact form on the new manifold that agrees with the old contact form outside the surgery domain and is a primitive of $d\lambda$. Note that $\frac{\partial}{\partial r} c$ is zero at $r = 0$, at $r = \pm\eta$, and at $w = \pm\epsilon$. Thus, the vector field R defined by being F outside the surgery domain and being $\frac{1}{1 \pm \frac{\partial}{\partial r} c} F$ in each side of the surgery domain is a well-defined vector field on the new manifold N. Since $(\lambda \pm dc)(R) \equiv 1$ and $\iota_R d\lambda = 0$, R is the Reeb vector field of the new contact form. In other words, the flow ψ_t obtained from the surgery is a time-change of the contact flow induced by R. Even more so, we notice that this is a canonical time-change. Therefore, ψ_t itself is a contact flow by proposition 1.28. We have verified:

Proposition 3.18. *For a specific choice of surgery annulus (namely, one with Legendrian knot and with $\frac{\partial}{\partial w} \in E^s \oplus E^u$ at $r = 0$) and choice of twist map, Anosov Dehn surgery preserves Anosov stable Hamiltonian structures of HS-contact type.*

Example: Handel and Thurston Extended

Geodesic flows on unit tangent bundles are of contact type. Let us review the example of Handel and Thurston in light of the adaption proposed by Foulon and Hasselblatt. Recall that the surgery chart is given by

$$\Lambda \to U\Sigma,\ (r, s, w) \mapsto \left(\exp_{c(s)}\left(rl_0 e^{i(\nu(w)+\pi/2)}\dot{c}(s)\right), \frac{\partial}{\partial r}\exp_{c(s)}\left(rl_0 e^{i(\nu(w)+\pi/2)}\dot{c}(s)\right)\right),$$

where c is a simple closed geodesic and $w = \frac{1}{l_0}\cos(\nu + \pi/2)$. The contact form whose Reeb flow is the geodesic flow is the canonical 1-form $(\lambda_0)_{(x,u)}(X) = \langle u, X_H\rangle$. In particular, the surgery knot $\gamma(s) = (c(s), l_0 J\dot{c}(s))$ is Legendrian because

$$(\lambda_0)_{\gamma(s)}(\dot{\gamma}(s)) = l_0\langle J\dot{c}(s), \dot{c}(s)\rangle = 0.$$

Moreover, the vector field $\frac{\partial}{\partial \nu}$ at $r = 0$ is given by $(0, -l_0 e^{i\nu}\dot{c})$, which lies in $E^s \oplus E^u$ by theorem 2.6. We only need to verify that $w = \frac{1}{l_0}\cos(\nu + \pi/2)$ is the coordinate transformation obtained from adapting the surgery annulus to the Anosov Hamiltonian structure $d\lambda_0$. Then lemma 3.16 implies that in (r, s, w)-coordinates we have $\lambda_0 = dr + w ds$. Since c is a geodesic and, hence, $\nabla_s \dot{c} \equiv 0$, the vector field $\frac{\partial}{\partial S}$ (where S is the knot coordinate before the change of coordinates) at $r = 0$ is given by $(\dot{c}(S), 0)$. The proof of lemma 3.16 revealed that the transformation is given by

$$w(S,\nu) = \lambda_0\left(\frac{\partial}{\partial S}\right)\Big|_{(0,S,\nu)} = l_0\langle e^{i(\nu+\pi/2)}\dot{c}(S), \dot{c}(S)\rangle = l_0|\dot{c}|^2\cos(\nu + \pi/2) = w.$$

Thus, the adapted surgery is applicable and yields a contact flow on the new manifold. In addition, if we started with a separating geodesic, then the new flow is not algebraic by theorem 3.12. This is how Foulon and Hasselblatt concluded:

Theorem 3.19. *There exist non-algebraic contact Anosov flows in dimension three.*

Remark 3.20. *This is remarkable because a contact Anosov flow with smooth Anosov splitting is an algebraic flow ([BFL92, Thrm. 1]). Hence, the above theorem says something about the strength of the hypothesis of having a smooth splitting.*

Example: Suspensions

In this section, we investigate the second class of Anosov stable Hamiltonian structures given by suspensions. We discussed in proposition 1.35 when these examples arise. Let an Anosov stable Hamiltonian structure (M, Ω) admitting a closed stabilizing 1-form be given and denote by λ the stabilizing 1-form obtained by pulling back the canonical volume form on S^1 as in proposition 1.35. As an analogue to the contact surgery, we replace the use of Legendrian knots by ones that live in a single fiber. Similarly, we again assume that the vector field $\frac{\partial}{\partial \nu}$ is contained in $E^s \oplus E^u$ at $r = 0$. As before, we adapt the surgery annulus to the Anosov Hamiltonian structure Ω.

Lemma 3.21. *In the surgery domain, $d\lambda = dr$.*

Proof. Since λ is ϕ_t-invariant, we must have $\ker(\lambda) = E^s \oplus E^u$. As before, $E^s \oplus E^u$ is always the tangent bundle of the fibers of the suspension structure. Therefore,

$$\lambda\Big(\frac{\partial}{\partial s}\Big)\Big|_{(0,s,0)} \equiv 0 \quad \text{and} \quad \lambda\Big(\frac{\partial}{\partial w}\Big)\Big|_{(0,s,w)} \equiv 0$$

by assumption. Suppose λ has local expression $dr + f_1(r,s,w)ds + f_2(r,s,w)dw$. That $d\lambda$ is closed is equivalent to f_1 and f_2 not depending on r as well as having $\frac{\partial}{\partial w}f_1 = \frac{\partial}{\partial s}f_2$. Thus, $f_2 \equiv 0$, which implies that f_1 depends only on s, which in turn implies $f_1 \equiv 0$. $\square$

In particular, λ is preserved by Anosov Dehn surgery in such a surgery domain. It follows that the surgery not only preserves stability of the Anosov Hamiltonian structure (M, Ω) but also a stabilizing 1-form. This is in contrast to the contact case, where the contact property was only qualitatively preserved but realized by a newly constructed form (even the underlying contact structure changed). Moreover, this time we have no need for any additional requirements on the twist map.

Proposition 3.22. *For a specific choice of surgery annulus (namely, one with the knot being contained in a single fiber and with $\frac{\partial}{\partial w} \in E^s \oplus E^u$ at $r = 0$), Anosov Dehn surgery preserves Anosov stable Hamiltonian structures of suspension type including the preservation of a stabilizing 1-form.*

Having specialized the surgery in both the HS-contact and the suspension cases, we conclude with theorem 1.36:

Theorem 3.23. *For a specific choice of surgery annulus and twist map, Anosov Dehn surgery preserves Anosov stable Hamiltonian structures.*

3.4 Surgery for Virtually Contact Structures

By remark 3.13, we can call a Hamiltonian structure *algebraic* if some (any) of its Reeb flows are algebraic. As a consequence of the Foulon-Hasselblatt surgery, we found non-algebraic HS-contact Anosov Hamiltonian structures in dimension three. A natural extension of this question is: are there non-algebraic virtually contact (but not HS-contact) Anosov Hamiltonian structures in dimension three, as well? In this chapter, we mean to answer this question in the positive by adapting the previous surgery idea and applying it to magnetic flows.
First, let us briefly outline the idea of the adaption. Let (M, Ω) be a virtually contact Anosov Hamiltonian structure with a Reeb flow ϕ_t. Recall that this involves the existence of a cover $\hat{M} \to M$ and a primitive λ of the lift $\hat{\Omega}$ with $||\lambda||_\infty < \infty$ and $\inf_{x\in\hat{M}} |\lambda(\hat{F}(x))| > 0$ in some (any) lifted metric. We refer to such λ as a *virtual contact form.* We discussed how to adapt Anosov Dehn surgery to preserve Anosov Hamiltonian structures. To show that the new Hamiltonian structure is virtually contact we need to find a suitable contact form in a cover of the new manifold N. However, if we perform the surgery on M without any additional attention, then afterwards we do not really know what a cover of N may look like. Without knowing what a cover looks like, it is even less feasible to construct an appropriate virtual contact form. If we can perform an analogous surgery in a cover $\hat{M}$ of the original manifold, then we get a new manifold $\hat{N}$ from $\hat{M}$, which will cover N. This way, we can keep track of what happens to the virtual contact form we started with and, as in the Foulon-Hasselblatt surgery, rebuild a new contact form on $\hat{N}$. By doing all these steps with proper care, this new contact form will be a virtual contact form for the new Hamiltonian structure on N.
This concept requires the existence of virtually contact Anosov Hamiltonian structures that are not HS-contact to begin with. The existence of such is asserted by various examples of magnetic flows, namely magnetic flows coming from a metric of non-constant curvature, see corollary 2.44. In order to achieve the "proper care" mentioned above, we will work with a very specific class of these flows for which we can do some computations explicitly. For an arbitrary virtually contact Anosov Hamiltonian structure, we have, in general, insufficient information to carry out several steps performed below. The flow we will work with is a magnetic flow that is only a slight perturbation of a geodesic flow, which is achieved by considering a large energy level. Moreover, to perform explicit calculations, we will make a very slight perturbation of a constant curvature metric so that the Anosov splitting of the flow resembles the one described in theorem 2.9.

Preparing the Surgery Chart "Downstairs"

Let (Σ, g) be an oriented closed surface of strictly negative but non-constant curvature. Recall the twisted symplectic form $\Omega = \omega_0 + \pi^*\Omega_{\text{area}}$ on the energy level S_k, where $\pi\colon S_k \to \Sigma$, and let ϕ_t be the corresponding magnetic flow. As mentioned above, corollary 2.44 provides us with a suitable class of Hamiltonian structures: for any large energy level, (S_k, Ω) is Anosov and virtually contact, but not HS-contact. We need a knot along which we can perform surgery. In the spirit of wanting to do a surgery similar to Handel and Thurston's, let c be a either closed geodesic or a closed magnetic geodesic on Σ of energy k and consider the vector field $J\dot{c}$ along c. Since the curvature will be taken to be "almost constant", a geodesic and a magnetic geodesic differ only slightly. The existence of a closed magnetic geodesic in an arbitrary homotopy class for any energy level $k > c_u$ is always given (see, for instance, [Mer10, Thrm. 1.1]). The latter condition is automatically satisfied by proposition 2.43.

Lemma 3.24. *If the curvature is "almost constant" (in the sense that the difference between the minimal and maximal curvature is small), then $\gamma = (c, J\dot{c})$ is an E-transverse knot for ϕ_t.*

Proof. We observe that using the cone criterion to verify structural stability of Anosov flows simultaneously asserts that the Anosov splitting gets perturbed only little when we slightly perturb a given Anosov flow. Hence, if the curvature is "almost constant", then the Anosov splitting of our magnetic flow resembles the one established in theorem 2.9. Let g_0 denote the Riemannian metric on Σ whose curvature is constantly K_0, where $K_0 < 0$ is the constant for which $\max_{x\in\Sigma} |K(x) - K_0|$ is small. Denote by J_0 the canonical almost complex structure of g_0. Let c_0 denote the closed curve in the homotopy class of c that is a geodesic or a magnetic geodesic with respect to g_0. Take $k > -\frac{1}{2K_0}$ as required in theorem 2.9. For the Anosov splitting given in said theorem, we have: $(c_0, J_0\dot{c}_0)$ is E-transverse for the magnetic flow coming from g_0 if $k \neq -\frac{1}{K_0}$ (proved below). In particular, if we take a large energy level and take our original curvature such that $(c, J\dot{c})$ is only a slight perturbation of $(c_0, J_0\dot{c}_0)$, then the lemma follows. Asserting the condition for $\gamma_0 = (c_0, J_0\dot{c}_0)$ to be E-transverse is a simple system of linear equations. Let (x, v) denote the point $(c_0(s), J_0\dot{c}_0(s))$. Then $\dot{\gamma}_0(s) = (-J_0v, 0)$ or $\dot{\gamma}_0(s) = (-J_0v, J_0v)$ depending on whether c_0 is a geodesic or a magnetic geodesic. Let γ_0^1 denote the former case and γ_0^2 the latter so that $\dot{\gamma}_0^i = (-J_0v, \delta_{i2}J_0v)$. By theorem 2.9, there are real numbers a_i, b_i, d_i such that at (x, v) we have

$$
\begin{aligned}
\dot{\gamma}_0^i &= a_i\left(\sqrt{-(2kK_0+1)}H - V_{1,k}\right) + b_i\left(\sqrt{-(2kK_0+1)}H + V_{1,k}\right) + d_iF \\
&\iff \\
-J_0v &= a_i\left(\sqrt{-(2kK_0+1)}J_0v - v\right) + b_i\left(\sqrt{-(2kK_0+1)}J_0v + v\right) + d_iv, \\
\delta_{i2}J_0v &= a_i 2kK_0J_0v - b_i 2kK_0J_0v + d_iJ_0v \\
&\iff \\
0 &= -a_i + b_i + d_i, \\
-1 &= \sqrt{-(2kK_0+1)}a_i + \sqrt{-(2kK_0+1)}b_i, \\
\delta_{i2} &= 2kK_0a_i - 2kK_0b_i + d_i.
\end{aligned}
$$

The knot γ_0^i is not E-transverse for the magnetic flow coming from g_0 only if the above system has a solution with either $a_i = 0$ or $b_i = 0$. Solving this system shows that $a_i = 0$ does not admit a solution for both $i = 1, 2$ whereas $b_i = 0$ only admits a solution if $i = 2$ and $k = -\frac{1}{K_0}$. □

Now that we have an E-transverse knot, we can construct a parametrized neighborhood as for the Anosov Dehn surgery. As with the chart for Handel and Thurston's surgery, we use the vertical vector field $V(x, v) = (0, Jv)$ as our local E-transverse vector field. E-transversality is ensured by theorem 2.10. Thus, we obtain a 2-dimensional neighborhood of γ parametrized by $(s, \nu) = \rho_\nu(\gamma(s)) = (c(s), e^{i\nu}J\dot{c}(s))$, where $\rho_t(x, v) = (x, e^{it}v)$ denotes the flow of V. As before, we can again apply the flow map for small times to get a 3-dimensional neighborhood parametrized by (r, s, ν)-coordinates on $(-\eta, \eta) \times S^1 \times (-\epsilon', \epsilon')$ such that $\frac{\partial}{\partial r}$ is the infinitesimal generator of the flow. To preserve the Hamiltonian structure Ω, we need to adapt the coordinates as discussed in the previous section.

Lemma 3.25. *Adapting the coordinates to the Hamiltonian structure Ω is done by $w(\nu) = -2k\cos(\nu + \frac{\pi}{2})$.*

Proof. Write Ω as $g d\nu \wedge ds$ in the chart for some smooth function g, which does not depend on the r-coordinate because Ω is closed. Note that at $r = 0$ the vector fields $\frac{\partial}{\partial S}$ and $\frac{\partial}{\partial \nu}$ are given by $(\dot{c}(S), \star)$ and $(0, -e^{i\nu}\dot{c}(S))$, respectively, where S denotes the knot coordinate before the change of coordinates and $\star$ denotes either 0 or $-e^{i\nu}\dot{c}(S)$ depending on whether c is a geodesic or a magnetic geodesic. Thus, we obtain

$$g(S,\nu) = \Omega\left(\frac{\partial}{\partial\nu}, \frac{\partial}{\partial S}\right)\Big|_{(0,S,\nu)} = -\langle -e^{i\nu}\dot{c}(S), \dot{c}(S)\rangle = |\dot{c}|^2\cos(\nu) = 2k\cos(\nu).$$

The transformation becomes $w(S,\nu) = \int_0^\nu g(S,\nu')d\nu' = 2k\sin(\nu)$. □

Denote the domain of the w-coordinate by $(-\epsilon,\epsilon)$ and the domain of the triple (r,s,w) by Λ. In summary,

$$\Lambda \hookrightarrow S_k,\ \ (r,s,w) \mapsto \phi_r\big(c(s), e^{i(\nu(w)+\frac{\pi}{2})}\dot{c}(s)\big).$$

We will switch freely between regarding Λ as a parameter space or a subset of S_k and hope this causes no confusion. Having set up a suitable parametrized neighborhood "downstairs", we turn our attention to a cover "upstairs", where the contact part of the surgery will take place.

Preparing the Surgery Chart "Upstairs"

Let $\tilde{\Sigma}$ denote the universal cover of Σ with covering map Π. Then $\tilde{S}_k \subset T\tilde{\Sigma}$ is a cover of S_k and we denote its covering map by $\tilde{\Pi}_k$. Since $k > c_u$, the cover $\tilde{S}_k$ admits a virtual contact form for (S_k,Ω). However, we would actually like to work in a smaller cover for reasons that become clear later. But first, let us exploit the information we have for $\tilde{S}_k$. Fix any lifted metric on $\tilde{S}_k$. There exists a 1-form Θ on $\tilde{\Sigma}$ that is bounded by $\frac{1}{2}||\Theta||_\infty^2 < k$ and is a primitive of $\Pi^*\Omega_{\text{area}}$. In fact, we can pick the primitive Θ more carefully. There is a deck transformation T of the cover $\tilde{\Sigma}\to\Sigma$ associated to the homotopy class $[c]$. Let Γ denote the infinite cyclic subgroup of $\pi_1(\Sigma,\star)$ generated by T. Then we may assume that Θ is Γ-invariant by proposition 2.35. To shorten notation, we abbreviate $\tilde{\Theta} = \tilde{\pi}^*\Theta$, $\tilde{\lambda}_0 = \tilde{\Pi}_k^*\lambda_0$ and $\tilde{\Omega} = \tilde{\Pi}_k^*\Omega$ so that all forms with a tilde on top live on $\tilde{S}_k$. Let $\tilde{\lambda}$ denote the 1-form on $\tilde{S}_k$ given by $-\tilde{\lambda}_0 + \tilde{\Theta}$. Then $\tilde{\lambda}$ is a virtual contact form for (S_k,Ω) because it is a primitive of $\tilde{\Omega}$, it is bounded, and it satisfies

$$\iota_{\tilde{F}}\tilde{\lambda} = -\tilde{\lambda}_0(\tilde{F}) + \tilde{\Theta}(\tilde{F}) \leq -2k + \sqrt{2k}||\Theta||_\infty < 0.$$

As mentioned above, we want to pass to a smaller cover. Consider the map $T\tilde{\Sigma}\to T\tilde{\Sigma}$ that sends (x,v) to $(T(x),(dT)_x(v))$. Since T is an element of $\mathrm{PSL}(2,\mathbb{C})$, it has determinant 1 so that this map restricts to a map $t\colon \tilde{S}_k \to \tilde{S}_k$. Since $\Pi\circ T = \Pi$, t clearly is a deck transformation of the cover $\tilde{\Pi}_k\colon \tilde{S}_k\to S_k$. Denote by $\tilde{\Gamma}$ the subgroup of deck transformations generated by t. Let $\hat{S}_k$ denote the quotient space $\tilde{S}_k/\tilde{\Gamma}$. Since deck transformations act properly discontinuously, $\hat{S}_k$ is a smooth manifold and, moreover, it is a cover of S_k. Denote its covering map by $\hat{\Pi}_k\colon \hat{S}_k\to S_k$. This will be the cover we want to work in for the surgery. Since $\tilde{F}$ and $\tilde{\lambda}_0$ are lifts of objects defined on S_k, they factor through $\tilde{\Gamma}$. Having picked Θ T-invariant, its lift also factors through because

$$t^*\tilde{\Theta} = (\tilde{\pi}\circ t)^*\Theta = (T\circ\tilde{\pi})^*\Theta = \tilde{\pi}^*\Theta = \tilde{\Theta}.$$

Thus, $\tilde{F}$, $\tilde{\lambda}_0$, $\tilde{\Omega}$, $\tilde{\Theta}$, and $\tilde{\lambda}$ descend to define $\hat{F}$, $\hat{\lambda}_0$, $\hat{\Omega}$, $\hat{\Theta}$, and $\hat{\lambda}$ on $\hat{S}_k$, respectively. Next, we construct the surgery chart "upstairs". Fix a lift $\hat{p}_0 \in \hat{S}_k$ of the point $p_0\in S_k$ given by the $(0,0,0)$-coordinate in Λ. By the lifting property of a cover, the map $(-\eta,\eta)\times\mathbb{R}\times(-\epsilon,\epsilon)\to S_k$, which is given by the (r,s,w)-coordinate but the s-coordinate read modulo 2π, has a unique lift to a map into $\hat{S}_k$ mapping the origin to $\hat{p}_0$. Denote the lifted coordinates by (r,s,w), as well, and let $\hat{\Lambda}$ denote the domain of the lifted coordinates (which must be either $(-\eta,\eta)\times\mathbb{R}\times(-\epsilon,\epsilon)$ or $(-\eta,\eta)\times S^1\times(-\epsilon,\epsilon)$). The map $\hat{\Lambda}\to\hat{S}_k,\ s\mapsto(0,s,0)$ is a concatenation of lifts of the knot γ and will be denoted by $\hat{\gamma}$.

Lemma 3.26. *The projection of $\hat{\gamma}$ onto γ is injective. Hence, $\hat{\Lambda} = (-\eta,\eta)\times S^1\times(-\epsilon,\epsilon) = \Lambda$ (as parameter spaces) and the projection of the lifted coordinates is a diffeomorphism.*

Proof. Similarly as above, we can lift the (r, s, w)-coordinates to (R, S, W)-coordinates on $\tilde{S}_k$ parametrized by the domain $(-\eta, \eta) \times \mathbb{R} \times (-\epsilon, \epsilon)$. Let $\tilde{\gamma}$ denote the concatenation of lifts of γ given by $S \mapsto (0, S, 0)$ and let $\tilde{c}$ denote the concatenation of lifts of c given by $\tilde{\pi} \circ \tilde{\gamma}$. Since T is the deck transformation associated to the homotopy class of c, we find $T(\tilde{c}(S)) = \tilde{c}(S + 2\pi)$ and $(dT)_{\tilde{c}(S)}(\dot{\tilde{c}}(S)) = \dot{\tilde{c}}(S + 2\pi)$. Thus, we find in (R, S, W)-coordinates

$$t(0, S, 0) = t(\tilde{\gamma}(S)) = \tilde{\gamma}(S + 2\pi) = (0, S + 2\pi, 0).$$

As the (R, S, W)-coordinates arose as lifts and t is a deck transformation, this implies $t(R, S, W) = (R, S + 2\pi, W)$. Thus, taking the quotient by $\tilde{\Gamma}$ means that we actually regard the S-coordinate modulo 2π again. The conclusion follows. □

Since $\hat{\Lambda} = \Lambda$, we will later be able to apply the same gluing map "upstairs" and "downstairs" such that the new object upstairs covers the new object downstairs. This is why we passed from the cover $\tilde{S}_k$ to the smaller one $\hat{S}_k$. For the former, the lifted chart is of the form $(-\eta, \eta) \times \mathbb{R} \times (-\epsilon, \epsilon)$ and twisting the s-coordinate by $2\pi q$ would not give rise to a new manifold. On the other hand, $\hat{S}_k$ is, in some sense, the smallest cover for which the lifted chart is of the form $(-\eta, \eta) \times S^1 \times (-\epsilon, \epsilon)$. However, in order for the entire new manifold $\hat{N}$ coming from $\hat{S}_k$ to cover N coming from S_k, we need to perform the surgery "upstairs" in every possible lift of Λ, not just the one we specified above by fixing $\hat{p}_0$. Fortunately, we can make sure this is possible.

Lemma 3.27. *Let $\hat{p}_1$ denote another lift of p_0. By lifting the coordinates to $\hat{S}_k$ so that the origin corresponds to $\hat{p}_1$ instead of $\hat{p}_0$, we get a new chart in $\hat{S}_k$ denoted $\hat{\Lambda}_1$. Then, up to shrinking ϵ and η if necessary, $\hat{\Lambda}$ and $\hat{\Lambda}_1$ (as subsets of $\hat{S}_k$) do not intersect. Moreover, the size of ϵ and η can be taken independently of $\hat{p}_0$ and $\hat{p}_1$.*

Proof. This only relies on the properties of a covering space. By definition, every point p in S_k has a neighborhood U_p such that $\hat{\Pi}_k^{-1}(U_p)$ is a disjoint union of sets diffeomorphic to U_p. Now take finitely many points $p_1, \ldots, p_N$ on γ such that $U_{p_1}, \ldots, U_{p_N}$ cover γ. Shrink ϵ and η so much that Λ is contained in the union $U_{p_1} \cup \cdots \cup U_{p_N}$. Let $\mathcal{U}$ denote the collection of the following sets V: V is a lift of U_{p_j} (for some $1 \leq j \leq N$) obtained by lifting p_j to a point lying on $\hat{\gamma}$. Define $\hat{\gamma}_1$ and $\mathcal{U}_1$ analogously. Now suppose for contradiction that $\hat{\Lambda}$ and $\hat{\Lambda}_1$ intersect in a point q. Then there is some V in $\mathcal{U}$ containing q. V projects to some U_{p_j}. One of the lifts of U_{p_j} in $\mathcal{U}_1$ must contain q, as well. But then we found two different preimages of U_{p_j} that are not disjoint, contradicting the covering property. □

By the lemma, we find a collection of disjoint lifts of Λ. This collection is countable because $\pi_1(S_k, \star)$ is countable and lifts of p_0 correspond to elements in $\pi_1(S_k, \star)$. Denote this collection by $\{\hat{\Lambda}_n\}_{n \geq 0}$. As they are disjoint, we can carry out the surgery in each lift simultaneously so that the resulting manifold $\hat{N}$ will cover N. However, before we get to the actual surgery, we first need to study the contact form $\hat{\lambda}$ in these lifted charts to prepare the contact part of the surgery.

Preparing the Contact Form "Upstairs"

Remark 3.28. *Recall from the Foulon-Hasselblatt surgery that the contact form λ was deformed to $(\mathrm{id} \times D)^* \lambda$ on one side of the surgery, and that we perturbed it to $\lambda - dc$ and $\lambda + dc$ with $(\mathrm{id} \times D)^*(\lambda - dc) = \lambda + dc$ to counter this shift. This worked because the local expressions of λ and dc contained no coefficient function g that depended on the s-coordinate. Otherwise, $(\mathrm{id} \times D)^*(\lambda - dc)$ would contain a coefficient function of the form $g \circ (\mathrm{id} \times D) = g(r, s + T(w), w)$ and would have no chance of gluing smoothly to $\lambda + dc$ due to the appearance of "$+T(w)$" in the argument.*

Let $\hat{\Lambda}$ denote any one of the lifted charts. Suppose $\hat{\lambda}$ is written as $f_0 dr + f_1 ds + f_2 dw$ in $\hat{\Lambda}$. As mentioned in the remark, the problem we are facing is that these coefficient functions depend non-trivially on the

s-coordinate. The idea is to change $\hat{\lambda}$ by adding an exact 1-form to get rid of the troublesome terms. However, we have to do so in a way that preserves the virtual contact property. Let us begin by analyzing $\hat{\lambda}$ in $\hat{\Lambda}$. Since $\frac{\partial}{\partial r}$ is a time-change of the Reeb vector field of $\hat{\lambda}$, we find $d\hat{\lambda}(\frac{\partial}{\partial r}) \equiv 0$ and, hence,

$$\frac{\partial}{\partial s} f_0 = \frac{\partial}{\partial r} f_1 \quad \text{and} \quad \frac{\partial}{\partial w} f_0 = \frac{\partial}{\partial r} f_2.$$

Denote by g_0 the map $\iota_{\hat{F}}\hat{\Theta}$ restricted to $\hat{\Lambda}$ so that we can write

$$f_0 = \iota_{\hat{F}}\hat{\lambda} = -\iota_{\hat{F}}\hat{\lambda}_0 + \iota_{\hat{F}}\hat{\Theta} = -2k + g_0.$$

Define a new function on $\hat{\Lambda}$ by $A(r,s,w) = \int_0^r g_0(r',s,w)dr'$. Then the above equations of partial derivatives yield

$$\frac{\partial}{\partial r} A = g_0, \quad \frac{\partial}{\partial s} A = f_1 - f_1(0,s,w), \quad \frac{\partial}{\partial w} A = f_2 - f_2(0,s,w).$$

Recall that the w-coordinate was constructed on integral flow lines of the vertical vector field V through γ so that $(0,s,w) = (c(s), e^{i(\nu(w)+\frac{\pi}{2})}\dot{c}(s))$. In particular, $\frac{\partial}{\partial w}$ has zero horizontal component at coordinates $(0,s,w)$. Therefore,

$$f_2(0,s,w) = \left[-\hat{\lambda}_0\left(\frac{\partial}{\partial w}\right) + \hat{\Theta}\left(\frac{\partial}{\partial w}\right)\right]\Big|_{(0,s,w)} \equiv 0.$$

Likewise, we can use $\frac{\partial}{\partial s}\big|_{(0,s,w)} = (\dot{c}, \star)$ (where $\star$ is either 0 or $-e^{i\nu}\dot{c}$ depending on whether c is a geodesic or a magnetic geodesic) to explicitly compute the function f_1 at these coordinates:

$$\begin{aligned} f_1(0,s,w) &= -(\lambda_0)_{\left(c, e^{i(\nu+\frac{\pi}{2})}\dot{c}\right)}((\dot{c},\star)) + \tilde{\Theta}_{\left(\tilde{c}, e^{i(\nu+\frac{\pi}{2})}\dot{\tilde{c}}\right)}((\dot{\tilde{c}},\star) \\ &= -\left\langle e^{i(\nu+\frac{\pi}{2})}\dot{c}, \dot{c}\right\rangle + \Theta_{\tilde{c}}(\dot{\tilde{c}}) \\ &= -||\dot{c}||^2 \cos\left(\nu + \frac{\pi}{2}\right) + \Theta_{\tilde{c}}(\dot{\tilde{c}}) \\ &= w + \Theta_{\tilde{c}}(\dot{\tilde{c}}). \end{aligned}$$

Then in $\hat{\Lambda}$ we have $\hat{\lambda} - dA = -2kdr + \left(w + \Theta_{\tilde{c}}(\dot{\tilde{c}})\right) ds$. Define another function in $\hat{\Lambda}$ by

$$B(r,s,w) = B(s) = \int_0^s \Theta_{\tilde{c}(s')}\left(\dot{\tilde{c}}(s')\right)ds'.$$

Then in $\hat{\Lambda}$ we get $\hat{\lambda} - d(A+B) = -2kdr + wds$, which looks pretty much like the form in the Foulon-Hasselblatt surgery (i.e. $dr + wds$) apart from the factor $-2k$. Being a constant factor, this does not bother us, though.

Remark 3.29. *The resemblance to the computations in the extended Handel and Thurston example for the geodesic flow is obvious. They correspond exactly to the case $\Theta = 0$.*

The forms dA and dB are only defined in $\hat{\Lambda}$, so $\hat{\lambda} - d(A+B)$ is not well-defined on $\hat{S}_k$. Let $\rho(r,w)$ be a bump function that is 0 outside $\hat{\Lambda}$ and 1 inside the smaller domain $\hat{\Lambda}^{sm} = (-\eta_{\mathrm{sm}}, \eta_{\mathrm{sm}}) \times S^1 \times (-\epsilon_{\mathrm{sm}}, \epsilon_{\mathrm{sm}})$. For small η_{sm}, we may pick ρ such that $|\partial_r \rho|$ is bounded by $2/\eta$.

Lemma 3.30. *Suppose we started with a high energy level (more precisely, $k > \left(3 + \frac{4\pi}{\eta}\right)^2 c_u$). Then the adjusted form $\hat{\lambda}_{\mathrm{adj}} = \hat{\lambda} - d(\rho(A+B))$ is a virtual contact form for (S_k, Ω).*

Proof. Certainly $d\hat{\lambda}_{\text{adj}} = d\hat{\lambda} = \hat{\Omega}$. Since $\hat{\lambda}$ is bounded and since A, B, dA, and dB only contain coefficient functions of $\hat{\lambda}$, the new form remains bounded, as well. We need to verify that $|\hat{\lambda}_{\text{adj}}|$ is bounded from below on $\text{span}\langle\hat{F}\rangle$. Outside $\hat{\Lambda}$, this is given as $\hat{\lambda}_{\text{adj}} = \hat{\lambda}$. We begin by estimating

$$\begin{aligned} |A|_\infty &\leq \eta |g_0|_\infty \leq \eta\sqrt{2k}||\Theta||_\infty, \\ |B|_\infty &\leq 2\pi||\Theta||_\infty|\dot{c}| = 2\pi\sqrt{2k}||\Theta||_\infty. \end{aligned}$$

Inside $\hat{\Lambda}$ we have $\hat{F} = \frac{\partial}{\partial r}$ and we further estimate

$$\begin{aligned} \hat{\lambda}_{\text{adj}}(\hat{F}) &= \hat{\lambda}(\hat{F}) - \rho\partial_r(A+B) - (\partial_r\rho)(A+B) \\ &= -2k + g_0 - \rho g_0 - (\partial_r\rho)(A+B) \\ &\leq -2k + |g_0|_\infty + |\partial_r\rho|_\infty(|A|_\infty + |B|_\infty) \\ &\leq -2k + \sqrt{2k}||\Theta||_\infty + \frac{2}{\eta}\Big(\eta\sqrt{2k}||\Theta||_\infty + 2\pi\sqrt{2k}||\Theta||_\infty\Big) \\ &= -2k + \sqrt{2k}||\Theta||_\infty\Big(3 + \frac{4\pi}{\eta}\Big) \end{aligned}$$

This last term is strictly smaller than 0 if $\frac{1}{2}||\Theta||_\infty^2 < k\big(3 + \frac{4\pi}{\eta}\big)^{-2}$, which is ensured when the right hand side is larger than the universal Mañé critical value. □

We performed the previous computation in one fixed lifted domain $\hat{\Lambda}$ of Λ. Now consider the collection $\{\hat{\Lambda}_n\}_{n\geq 0}$ of lifted domains. Let A_n and B_n denote the corresponding functions for the lift $\hat{\Lambda}_n$ and pick a bump function ρ_n for $\hat{\Lambda}_n$ as above. Then the new form $\lambda = \hat{\lambda} - \sum_{n\geq 0} d\big(\rho_n(A_n + B_n)\big)$ is a virtual contact form for (S_k, Ω) and inside each smaller domain $\hat{\Lambda}_n^{sm}$ it is given by the expression $\lambda = -2kdr + wds$.

The Surgery

The actual surgery works precisely as the Foulon-Hasselblatt one. Let $T\colon (-\epsilon_{sm}, \epsilon_{sm}) \to S^1$ be the same twist map as before and $D(s,w) = (s + T(w), w)$. Then we perform Anosov Dehn surgery in Λ^{sm} to get a new flow ψ_t on the new manifold N. We can conduct the same surgery in each lift $\hat{\Lambda}_n^{sm}$ simultaneously. Since we use the same gluing map, the resulting manifold $\hat{N}$ will be a cover of N and the new flow "upstairs" covers the new flow "downstairs". In every lifted domain $\hat{\Lambda}_n^{sm}$ we can define the same correction function $c_n(r,w)$ as for the Foulon-Hasselblatt surgery and perturb λ to $\lambda - dc_n$ and to $\lambda + dc_n$ on each side of the surgery, respectively. Since each c_n vanishes outside $\hat{\Lambda}_n^{sm}$, we can do this simultaneously in every domain, as well. This way, we get a contact form λ_{new} on $\hat{N}$. Since λ is a virtual contact form for (S_k, Ω), there is a non-zero function R on $\hat{S}_k$ bounded from above and below such that the time-change $R\hat{F}$ is the Reeb vector field of λ. Note that $R \equiv -\frac{1}{2k}$ inside each $\hat{\Lambda}_n^{sm}$. Then, $R_{\text{new}} = R \cdot \prod_{n\geq 0} 1/(1 \pm \frac{\partial}{\partial r}c_n)$ is a well-defined non-zero function on $\hat{N}$ bounded from above and below, where $\pm$ corresponds to either side of the surgery, as usual. By construction, $R_{\text{new}}\hat{F}$ is the Reeb vector field of λ_{new} and we conclude that λ_{new} is a virtual contact form for (N, Ω).

Since this is a special case of the general Anosov Dehn surgery, the new flow ψ_t is Anosov. If the knot c is both simple and separating, then the new flow is also non-algebraic by theorem 3.12 as the proof of this result only relied on topological properties of $U\Sigma \cong S_k$. Note that there are admissible knots that are simple and separating since we could have taken any closed geodesic. Thus, we constructed a non-algebraic virtually contact Anosov Hamiltonian structure (N, Ω). Of course, the Foulon-Hasselblatt surgery also yields such structures since HS-contact implies virtually contact. However, we can verify that our new structure is not covered by their examples:

Theorem 3.31. *There exist non-algebraic virtually contact Anosov Hamiltonian structures in dimension three that are not HS-contact.*

To prove this theorem, we first inquire about the regularity of the associated 1-forms of Anosov flows.

Lemma 3.32. *Given an Anosov flow $\rho_t \colon M \to M$, the bundle $E^s \oplus E^u$ (and, hence, the associated 1-form of the flow) has the same regularity along orbits.*

Proof. Let $\Pi \colon E^s \oplus E^u \to M$ denote the restriction of the projection $TM \to M$ to $E^s \oplus E^u$. Abbreviate this vector bundle by E. Take an open cover U_α of M and local trivializations $\mathrm{tr}_\alpha \colon \Pi^{-1}(U_\alpha) \to U_\alpha \times V$ of E, where $V \cong \mathbb{R}^n$ and $n = \dim(E) = \dim(M) - 1$. We can equip $E^s \oplus E^u$ with a different bundle structure E' as follows. Set $U'_\alpha = \rho_{-t}(U_\alpha)$ so that, by ρ_t-invariance, $\Pi^{-1}(U'_\alpha) = d\rho_{-t} \circ \Pi^{-1}(U_\alpha)$. Then define $\mathrm{tr}'_\alpha \colon \Pi^{-1}(U'_\alpha) \to U'_\alpha \times V$ by $\mathrm{tr}'_\alpha = (\rho_{-t} \times \mathrm{id}_V) \circ \mathrm{tr}_\alpha \circ d\rho_t$. It is easily verified that this is a bundle structure. Moreover, the identity map $E^s \oplus E^u \to E^s \oplus E^u$ is a bundle isomorphism between E and E'. Suppose $p \in U_\alpha$ and $\rho_t(p) \in U_\beta$. Then local trivializations of E and E' above p are given by tr_α and $(\rho_{-t} \times \mathrm{id}_V) \circ \mathrm{tr}_\beta \circ d\rho_t$, respectively, from which we find that $E^s \oplus E^u$ has the same regularity at p and at $\rho_t(p)$. $\square$

Proof of theorem 3.31. All properties are proved except for not being HS-contact. We need to argue why no time-change of ψ_t can be a contact flow. Suppose there was a time-change ψ''_t that is contact. By lemma 3.15, there is a time-change ϕ'_t of ϕ_t so that doing ϕ'_t-surgery yields a flow ψ'_t that is smoothly conjugate to a constant time-change of ψ''_t. Since a constant time-change and a smooth conjugacy preserve the contact property of a flow, ψ'_t is contact. Let $\lambda_{\phi'}$ and $\lambda_{\psi'}$ denote the associated 1-forms. By remark 3.4, these two forms agree outside the surgery domain. The flow ϕ'_t is not contact by hypothesis because it is a Reeb flow of the Hamiltonian structure provided by corollary 2.44. Since we are definitely not in the suspension case, theorem 1.32 tells us that $\lambda_{\phi'}$ cannot be smooth everywhere. Let $\mathrm{Sing}(\phi')$ denote the set of points where $\lambda_{\phi'}$ is not smooth and likewise for any other flow. We assumed that ψ'_t is contact, so $\mathrm{Sing}(\psi')$ is empty. That $\lambda_{\phi'} = \lambda_{\psi'}$ outside the surgery domain implies $\mathrm{Sing}(\phi') \subset \Lambda$. Further, the previous lemma asserts $\phi'_t(\mathrm{Sing}(\phi')) \subset \mathrm{Sing}(\phi') \subset \Lambda$ for all times t. However, this contradicts Λ being a flow box. $\square$

A Appendix

A.1 The Tangent and Cotangent Bundle

We will show that for every Riemannian manifold (M, g) the unit tangent bundle admits a canonical contact form. But first, we recall the splitting of the tangent bundle into its horizontal and its vertical part. Let us use the notation $\langle \cdot, \cdot \rangle$ for g. From now on, we will identify TM and T^*M via the Riemannian metric, $TM \xrightarrow{\sim} T^*M$, $u \mapsto \langle u, \cdot \rangle$. Let ∇ be the Levi-Civita connection of g and $\pi\colon T^*M \to M$ the projection map. To ∇ there is an associated connection map κ defined as follows. Given $X \in T(T^*M)$, take a path $c(t) = (x(t), u(t))$ in T^*M with $\partial_t c(0) = X$. Then $\kappa\colon T(T^*M) \to T^*M$ maps X to $\nabla_t u(0)$. This map together with the projection map induces an isomorphism of vector bundles

$$T(T^*M) \xrightarrow{\sim} \pi^*(TM) \oplus \pi^*(T^*M), \ X \mapsto (d\pi(X), \kappa(X)).$$

We call $\pi^*(TM)$ the horizontal component of $T(T^*M)$, $\pi^*(T^*M)$ the vertical component, and write

$$X_H = d\pi(X), \quad X_V = \kappa(X), \quad \text{and} \quad X = (X_H, X_V).$$

We always implicitly assume this splitting when dealing with second tangent bundles. Using this splitting, we can define a Riemannian metric g_s on T^*M called the *Sasaki metric* by

$$g_s(X, Y) = \langle X_H, Y_H \rangle + \langle X_V, Y_V \rangle.$$

Moreover, we also get an almost complex structure J_g on T^*M via $J_g(X) = (-X_V, X_H)$. Our goal now is to find a 1-form λ_0 on T^*M with the following properties:

(1) $\omega_0 = -d\lambda_0$ is a symplectic form on T^*M,

(2) (g_s, ω_0, J_g) is a compatible triple, i.e. $\omega_0(\cdot, J_g \cdot) = g_s(\cdot, \cdot)$,

(3) the restriction of λ_0 to the unit cotangent bundle U^*M is a contact form.

Let us verify that the *Liouville 1-form* $(\lambda_0)_{(x,u)}(X) = \langle u, X_H \rangle$ does the job. Using the geodesic vector field on T^*M given by $G(x, u) = (u, 0)$ (neglecting the base-point), we can rewrite λ_0 as

$$\lambda_0(X) = \langle d\pi(G), d\pi(X) \rangle = g_s(G, X).$$

Note that the connection $\pi^*(\nabla)$ on the bundle $\pi^*(TM) \to T^*M$ clearly preserves the inner product $\langle \cdot, \cdot \rangle$ under parallel transport as it is just the pullback of ∇, so it is a Riemannian connection on $\pi^*(TM)$ with respect to $\langle \cdot, \cdot \rangle$. Since $d\pi(G)$ is simply the "identity section" of $\pi^*(TM)$ in the sense that it maps $(x, u) \in TM$ to $((x, u); (x, u)) \in \pi^*(TM)$, we have, by definition of κ, $(\pi^*\nabla)_X(d\pi(G)) = \kappa(X)$ for any vector field X on T^*M. Furthermore, $\pi^*\nabla$ surely is torsion-free since ∇ is. We can now deduce a nice representation for $d\lambda_0$, where we use the notation $\nabla' = \pi^*\nabla$:

$$\begin{aligned}
d\lambda_0(X, Y) &= \mathcal{L}_X\left(\lambda(Y)\right) - \mathcal{L}_Y\left(\lambda(X)\right) + \lambda([X, Y]) \\
&= \mathcal{L}_X\left(\langle d\pi(G), d\pi(Y) \rangle\right) - \mathcal{L}_Y\left(\langle d\pi(G), d\pi(X) \rangle\right) + \langle d\pi(G), d\pi([X, Y]) \rangle \\
&= \langle \nabla'_X d\pi(G), d\pi(Y) \rangle + \langle d\pi(G), \nabla'_X d\pi(Y) \rangle \\
&\quad - \langle \nabla'_Y d\pi(G), d\pi(X) \rangle - \langle d\pi(G), \nabla'_Y d\pi(X) \rangle + \langle d\pi(G), d\pi([X, Y]) \rangle \\
&= \langle \kappa(X), d\pi(Y) \rangle - \langle \kappa(Y), d\pi(X) \rangle \\
&\quad + \langle d\pi(G), \underbrace{\nabla'_X d\pi(Y) - \nabla'_Y d\pi(X) + d\pi([X, Y])}_{=0} \rangle \\
&= \langle \kappa(X), d\pi(Y) \rangle - \langle \kappa(Y), d\pi(X) \rangle.
\end{aligned}$$

Thus, ω_0 takes the simple form

$$\omega_0(X,Y) = \langle X_H, Y_V\rangle - \langle X_V, Y_H\rangle,$$

which is clearly non-degenerate. Further, in this form, the equation $\omega_0(\cdot, J_g\cdot) = g_s(\cdot,\cdot)$ becomes evident. Lastly, $\lambda_0 \wedge \omega_0^{n-1}$ never vanishes, which can be seen as follows: Take a g-orthonormal basis $v_0, \dots, v_{n-1}$ of T_xM with $v_0 = u$. Then $T_{(x,u)}UM$ is spanned by the vectors $X_j = (v_j, 0)$, $0 \le j \le n-1$ and $Y_j = (0, v_j)$, $1 \le j \le n-1$. By orthonormality, it holds that $\omega_0(X_j, Y_k) = \delta_{j,k}$ as well as $\omega_0(X_j, X_k) = 0 = \omega_0(Y_j, Y_k)$ and, hence, a quick induction step shows $\omega_0^k(X_{j_1}, Y_{j_1}, \dots, X_{j_k}, Y_{j_k}) = k!$, for any $1 \le j_1, \dots, j_k \le n-1$. Thus,

$$\lambda_0 \wedge \omega_0^{n-1}(X_0, X_1, Y_1, \dots, X_{n-1}, Y_{n-1}) = (n-1)!,$$

so the restriction of λ_0 to the unit tangent bundle is indeed a contact form.

Remark A.1. *We can also define λ_0 on T^*M by $(\lambda_0)_{(x,u)}(X) = u(d\pi_{(x,u)}(X))$. Therefore, there is a canonical 1-form on T^*M independent of the Riemannian metric. However, we need an explicit Riemannian metric to translate this 1-form to TM. In local coordinates $(x_1, \dots, x_n, u_1, \dots, u_n)$ on T^*M, λ_0 and ω_0 become $\lambda_0 = \sum_{k=1}^n u_k dx_k$ and $\omega_0 = \sum_{k=1}^n dx_k \wedge du_k$, which also shows non-degeneracy.*

We would like to briefly recall that graphs of 1-forms are Lagrangian submanifolds of the cotangent bundle equipped with the standard symplectic form if and only if the 1-form is closed. Thus, suppose N is the graph of a 1-form μ on M. Let s denote the actual map $M \to T^*M,\ x \mapsto (x, \mu_x)$ whose graph is the submanifold N. Then we compute

$$(s^*\lambda_0)_x(u) = \mu_x(d\pi \circ ds(u)) = \mu_x(u).$$

Therefore, we find that N is a Lagrangian submanifold of (TM, ω_0) if and only if

$$0 = s^*\omega_0 = -s^*d\lambda_0 = -d\mu.$$

A.2 Dehn Surgery

A Dehn surgery is the process of cutting a solid 3-torus out of a manifold and gluing it back in a different way. The gluing is specified by a pair of integers (p,q) that represent the homology class in $H_1(T^2) \cong \mathbb{Z} \oplus \mathbb{Z}$ to which we glue the meridian and the (fixed) longitude of the boundary T^2 of the solid torus. Up to isotopy, this uniquely determines the manifold obtained from the Dehn surgery. Even more so, the surgery coefficient $p/q \in \mathbb{Q} \cup \{\infty\}$ determines the surgery up to isotopy. The picture simplifies for $p/q \in \mathbb{N}$. Then instead of cutting and gluing a solid torus, we can imagine cutting the given torus along an annulus and gluing the two annuli back together with a twist. This twist leaves one boundary component of the annulus fixed whereas it rotates the other boundary component p/q-times. Here is how the surgery is technically conducted in the case $p/q \in \mathbb{N}$.

Denote the ambient manifold by M and the annulus by A. Pick a tubular neighborhood of A in M denoted by $\Psi\colon (-\eta, \eta) \times A = \Lambda \hookrightarrow M$. Let V and W denote the images of the restriction of Ψ to $(-\eta, 0] \times A$ and $[0, \eta) \times A$, respectively. Next, cut M open along the annulus, and consider V and W as subsets of M_{cut}, each with a boundary diffeomorphic to A. Suppose we are also given a diffeomorphism D of the annulus, the gluing map. We require D to smoothly tend to the identity map as we near the boundary of the annulus. To obtain a Dehn surgery with coefficient $p/q \in \mathbb{N}$, the gluing map D is taken so that it fixes one boundary component of A point-wise and rotates the other boundary component p/q times. Let us introduce the function

$$\tilde{\Phi}\colon V \sqcup W \to \Lambda,\ \tilde{\Phi}(x) = \begin{cases} \Psi^{-1}(x), & \text{if } x \in V, \\ \left(\mathrm{id} \times D^{-1}\right) \circ \Psi^{-1}(x), & \text{if } x \in W. \end{cases}$$

This map will be used to write down an atlas for the smooth structure on the glued manifold. The gluing is up next. Define an equivalence relation on M_{cut} by identifying

$$\partial V \ni x \sim \Psi\left(0, D \circ \mathrm{pr}_A \circ \Psi^{-1}(x)\right) \in \partial W,$$

where pr_A is the projection of Λ to A. Points outside ∂V and ∂W are untouched. This just means that we identify x and $D(x)$ on A, but working in the cut tubular neighborhood. Let $\pi\colon M_{\text{cut}} \to N = M_{\text{cut}}/\sim$ denote the corresponding quotient map. Since for $x \in \partial V$

$$\tilde{\Phi}\left(\Psi\left(0, D \circ \mathrm{pr}_A \circ \Psi^{-1}(x)\right)\right) = \left(0, \mathrm{pr}_A \circ \Psi^{-1}(x)\right) = \Psi^{-1}(x) = \tilde{\Phi}(x),$$

the map $\tilde{\Phi}$ descends to a function $\Phi\colon \pi(V \sqcup W) \to \Lambda$ on the quotient, which is actually a homeomorphism. A smooth atlas on N that agrees with the old smooth structure away from the surgery is now given as follows: take any chart (U, ϕ) on M that does not intersect A and add the chart $(\pi(U), \phi \circ \pi^{-1}|_{\pi(U)})$; given a chart (U, ϕ) on M meeting the annulus, add a chart for N by

$$\left((\Psi \circ \Phi)^{-1}(U),\ \phi \circ \Psi \circ \Phi\Big|_{(\Psi\circ\Phi)^{-1}(U)}\right).$$

Since D is the identity at the boundary of A, this structure is compatible with the old structure outside the annulus. Note that the inverse of Φ gives us a smooth parametrization of a tubular neighborhood of the glued region,

$$\Phi^{-1}\colon \Lambda \to \pi(V \sqcup W),\ (r, y) \mapsto \begin{cases} \Psi(r, y), & \text{if } r \geq 0, \\ \Psi \circ (\mathrm{id} \times D)(r, y), & \text{if } r < 0. \end{cases}$$

Using this parametrization together with the observation that π maps $M \setminus A$ identically onto $N \setminus A$, we can check whether any interesting objects on M are preserved by the surgery. Immediately, we can check that the vector field $\frac{\partial}{\partial r}$ remains unchanged since $d\left(\Phi \circ \pi \circ \Psi\right)\left(\frac{\partial}{\partial r}\right) = \frac{\partial}{\partial r}$ on $\{r \neq 0\}$. We also have

$$\Psi^{-1} \circ \pi^{-1} \circ \Phi^{-1}(r, y) = \begin{cases} (r, y), & \text{if } r > 0, \\ (r, D(y)), & \text{if } r < 0, \end{cases}$$

so that the pullback of a differential form λ on Λ under this map is

$$\left(\Psi^{-1} \circ \pi^{-1} \circ \Phi^{-1}\right)^* \lambda = \begin{cases} \lambda, & \text{if } r > 0, \\ (\mathrm{id} \times D)^* \lambda, & \text{if } r < 0, \end{cases}$$

Hence, a differential form λ on M continues to describe a well-defined differential form on N if its local expression $\Psi^*\lambda$ in Λ is invariant under $(\mathrm{id} \times D)$.

We note that we had to fix a tubular neighborhood of A to conduct the surgery. Let us investigate what happens when we use a different tubular neighborhood $\Psi'\colon (-\eta', \eta') \times A = \Lambda' \hookrightarrow M$. We may replace Λ and Λ' so that $\Psi(\Lambda) = \Psi'(\Lambda')$. Denote the change of coordinates map by $C = \Psi'^{-1} \circ \Psi\colon \Lambda \to \Lambda'$. Since we are dealing with tubular neighborhoods, C is of the form $C(r, s, w) = (C_1(r, s, w), s, w)$. Now define Φ' analogous to Φ above. The resulting manifold N' is the same topological space as N because Ψ and Ψ' are the same when restricted to $\{0\} \times A$. However, their smooth structure usually does not agree for otherwise

$$\begin{aligned} \Phi' \circ \Phi^{-1}\colon \Lambda \to \Lambda',\ (r, s, w) &\mapsto \begin{cases} C(r, s, w), & \text{if } r > 0, \\ \left(\mathrm{id} \times D^{-1}\right) \circ C \circ (\mathrm{id} \times D)\,(r, s, w), & \text{if } r < 0 \end{cases} \\ &= \begin{cases} (C_1(r, s, w), s, w), & \text{if } r > 0, \\ (C_1(r, D(s, w)), s, w), & \text{if } r < 0 \end{cases} \end{aligned}$$

would be a diffeomorphism, but it is not even continuous if C_1 is not D-invariant in the (s, w)-coordinates. However, N and N' are isotopic smooth structures as can be seen from creating a smooth structure N_t interpolating N and N' by taking an isotopy from C_1 to the identity. Furthermore, any diffeomorphism $h_{\text{loc}}\colon \Lambda \to \Lambda'$ that equals C near the boundary of Λ gives rise to a global diffeomorphism

$$h\colon N \to N',\ h(x) = \begin{cases} x, & \text{if } x \notin \Phi^{-1}(\Lambda), \\ \Phi'^{-1} \circ h_{\text{loc}} \circ \Phi(x), & \text{if } x \in \Phi^{-1}(\Lambda). \end{cases}$$

Taking h_{loc} to be of the form $h_{\text{loc}}(r, s, w) = (h_1(r, s, w), s, w)$, this diffeomorphism sends the vector field $\frac{\partial}{\partial r}$ to $\left(\frac{\partial}{\partial r} h_1\right) \frac{\partial}{\partial r'}$.

A.3 A Theorem by Egorov

In the proof of lemma 1.31, we need to use a well-known result from measure theory by Egorov. Because the version we need is formulated slightly differently than the one usually encountered, we provide a proof.

Theorem A.2 (Egorov)**.** *Suppose μ is a finite Radon measure on a subset Ω of euclidean space $\mathbb{R}^N$ and $f_n\colon \Omega \to \overline{\mathbb{R}}$ are μ-measurable functions. Further, suppose that point-wise $f_n(x) \to \infty$ on a set $R \subset \Omega$ of positive measure $\mu(R) > \delta > 0$. Then there exists a compact set $K \subset R$ with $\mu(K) > \delta$ on which $f_n \to \infty$ uniformly, i.e. $\inf_{x\in K} f_n(x) \to \infty$ as $n \to \infty$.*

Proof. Consider the sets $C_{i,j} = R \cap \bigcup_{k=j}^{\infty} \{x \in \Omega \mid f_n(x) \leq 2^i\}$. These are clearly μ-measurable and satisfy

$$\lim_{j\to\infty} \mu(C_{i,j}) = \mu\Big(\bigcap_{j=1}^{\infty} C_{i,j}\Big) = \mu(\emptyset) = 0,$$

by hypothesis. Take $\epsilon < \mu(R) - \delta$. Then every $i \in \mathbb{N}$ admits some $k(i) \in \mathbb{N}$ with $\mu(C_{i,k(i)}) < \epsilon/2^{i+1}$. Define $A = R \setminus \bigcup_{i=1}^{\infty} C_{i,k(i)}$ so that $\mu(R \setminus A) < \epsilon/2$. It holds that $\inf_{x\in A} f_n(x) > 2^i$ for $n \geq k(i)$ by construction. Since μ is a Radon measure, there exists a compact set $K \subset A$ with $\mu(A \setminus K) < \epsilon/2$. In particular,

$$\mu(K) = \mu(R) - \mu(R \setminus A) - \mu(A \setminus K) > \mu(R) - \epsilon > \delta.$$

□

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