An Introduction to Complex Dynamics and the Mandelbrot Set

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Abstract: We first show that the Julia set of a holomorphic map on the Riemann sphere is the closure of the repelling periodic points and discuss Sullivan's classification of the Fatou set. We then prove that the Julia set of $z \to z^d$ is either connected or totally disconnected and introduce the Mandelbrot set, of which we thereafter show connectedness. We proceed by using polynomial-like maps to show the existence of quasi-conformal copies of the Mandelbrot set inside itself. Lastly, by using holomorphic motions, we prove that the boundary of the Mandelbrot set has Hausdorff dimension two.

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1 Introduction

The simplest of maps such as polynomials of low degree can exhibit some extraordinary dynamics. For example, it is well known that the polynomial $z \to z^2$ restricted to the unit circle, commonly known as the doubling map, admits chaotic behavior. By considering a bigger picture, we can get much more fascinating results than in one dimension. We mean to give a broad introduction to the subject of holomorphic dynamics on the Riemann sphere. That is, the system in consideration will always be a holomorphic map $R\colon \overline{\mathbb{C}} \to \overline{\mathbb{C}}$. To investigate the above example in a more general way, Fatou and Julia independently introduced what we now call the *Fatou* and the *Julia set* of a given system. Roughly speaking, these represent the regions on the sphere where we have simple and exotic dynamics, respectively.

In the first part of this paper we will investigate the nature of these sets. Our main discoveries will be that the Julia set is the closure of the repelling periodic points and the Fatou set reduces to finitely many periodic components, which are fully described by Sullivan's classification. We conclude that section by studying polynomials, for which we usually can get much more intricate results than for a general holomorphic map. Most importantly, we show that for the polynomial $z \to z^d + c$, the Julia set must either be connected or totally disconnected, and we give a sufficient and necessary condition for each.

In the second part we move on to the *Mandelbrot set* $\mathcal{M}$, defined as the set of parameters c for which the above polynomial has a connected Julia set. This is a parameter space and, in fact, in degree two it embodies all possible classes of polynomial dynamics. We begin that second part by proving some standard results on the topology of $\mathcal{M}$, for example, that it is connected and full. We proceed by studying hyperbolicity of $\mathcal{M}$, by which we actually study hyperbolicity of dynamical systems. The pitch will be a review of the two conjectures about local connectedness and density of hyperbolicity, which are open problems and are still objects of current research. In the end, we give a brief sketch of the material covered in the famous *Orsay notes*, written by Douady and Hubbard.

Chapters 4 and 5 are the highlights of this thesis. They both center around the self-similarity of $\mathcal{M}$, each using a different notion. In chapter 4 we use polynomial-like maps, which were introduced by Douady and Hubbard, to show the occurrence of quasi-conformal small copies of $\mathcal{M}$ at its boundary. In the last chapter, we prove that the Mandelbrot set is a fractal. Namely, we show that the Hausdorff dimension of the boundary $\partial\mathcal{M}$ is two. We first review the construction of the Hausdorff dimension and then introduce the tool of holomorphic motions to prove the main theorem.

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2 The Julia and the Fatou Sets

2.1 Rational Dynamics

Let $R\colon \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be a non-constant holomorphic map. Since holomorphic maps are open, $R(\overline{\mathbb{C}})$ is an open and compact subset of $\overline{\mathbb{C}}$ and R must be surjective. It is a standard result that R automatically is a rational map. This can be seen as follows. If R had infinitely many poles, then it would be constant ∞ by the identity principle. If it has a single pole at ∞, then the restriction to $\mathbb{C}$ is an entire function and $1/R(\frac{1}{z})$ has a removable singularity at 0. Hence, in this case, R must be a polynomial. Now suppose $z_1, \ldots, z_n$ are its poles, where we assume that ∞ is not one of them since otherwise we could consider $\frac{1}{R}$ instead, and that k_j is the order of the pole at z_j. Then

$$R(z) \cdot \prod_{1 \leq j \leq n} (z - z_j)^{k_j}$$

is a polynomial by the previous step. Thus, R is rational.
The degree of R is the maximum of the degrees of its enumerator and its denominator. If the degree is 1, then R is a Möbius transformation, whose dynamics are already understood entirely. That is why we always consider maps of degree $d \geq 2$. Let us clarify that whenever the variable d is used, it always denotes the degree of the map we are currently working with.
We will recall a few facts about the dynamics of such a map. For a more detailed exhibition of the material, the reader may consult Blanchard's survey article [3] and Carleson and Gamelin's textbook [5], which this and partially the next chapter is based upon.

Since the study of dynamics evolves around the study of orbits, let us begin by specifying a few types that admit certain behavior.

Definition 2.1. *Suppose that $R^n(p) = p$. The eigenvalue of the periodic orbit $\{p, R(p), \ldots, R^{n-1}(p)\}$ is*

$$\lambda_p = (R^n)'(p) = \prod_{0 \leq j \leq n-1} R'(R^j(p))$$

We say a periodic orbit is
1. *attractive if $0 < |\lambda_p| < 1$,*
2. *superattractive if $\lambda_p = 0$,*
3. *repelling if $|\lambda_p| > 1$,*
4. *neutral if $|\lambda_p| = 1$ and*
5. *parabolic if λ_p is a root of unity.*

The regions where the dynamics of a rational map are particularly easy to study are the ones close to an attractive periodic orbit since each nearby point converges to that orbit. In order to talk about such behavior in a rigorous way, we give such a region a name.

Definition 2.2. *Given a (super-)attractive fixed point p of R, the attractive basin, or stable basin*[1] *is*

$$W^s(p) = W^s(p, R) = \{z \in \overline{\mathbb{C}} \mid R^k(z) \to p \text{ as } k \to \infty\}$$

The immediate attractive basin $A(p)$ is the connected component of $W^s(p)$ containing p.
Given a (super-)attractive periodic orbit $\{p, R(p), \ldots, R^{n-1}(p)\}$, the attractive basin is

$$W^s(\mathrm{Orb}(p)) = \bigcup_{0 \leq j \leq n-1} W^s(R^j(p), R^n)$$

[1] Similarly, we can define the unstable basin $W^u(p) = \{R^{-k}(z) \to p\}$

The immediate attractive basin $A(\mathrm{Orb}(p))$ is the union of the connected components containing the $R^j(p)$, $0 \leq j \leq n-1$. We often just write $W^s(p)$ and $A(p)$ instead of $W^s(\mathrm{Orb}(p))$ and $A(\mathrm{Orb}(p))$.

One goal of this chapter is to get an idea of not just these, but all the regions where the dynamics behave nicely. To formalize the idea of nice behavior, the notion of equicontinuity comes in useful. Namely, we consider the family of iterates $(R^n)_{n\geq 0}$. Equicontinuity of this family describes the behavior of the orbits of R. For historical reasons, we use the theorem of Arzelà-Ascoli and write everything in terms of the notion of normality. However, in the proofs we sometimes freely change between considering normality and equicontinuity.

Definition 2.3. *The Fatou set of R, denoted by $F(R)$ or simply F, contains all the points in $\overline{\mathbb{C}}$ for which there exists an open neighborhood on which $(R^n)_{n\geq 0}$ is a normal family. The Julia set, denoted by $J(R)$ or J, is the complement of the Fatou set.*

The Fatou set is open by definition. Clearly, if an attractive periodic point p exists, then $(R^n)_{n\geq 0}$ is equicontinuous on $W^s(p)$ and so $W^s(p) \subset F$. But the existence is not necessary. In fact, the Fatou set can be empty. We will see a sufficient condition for this later on in corollary 2.10. On the other hand, the Julia set never is empty.

Proposition 2.4. *The Julia set is always non-empty.*

Proof. If the Julia set was empty, then $(R^n)_{n\geq 0}$ was normal on all of $\overline{\mathbb{C}}$. Let S denote a limit function of this family. Being holomorphic, S has some degree $d' < \infty$. On the other hand, each map R^n has degree d^n, contradicting convergence to S on a subsequence. □

Just like we think of the Fatou set as the region with nice dynamics, the Julia set represents the region, where R displays exotic, sometimes chaotic behavior (a condition for chaos will be given further below). In view of proposition 2.4 we can say that a rational map *always* exhibits some interesting dynamics. We prove below that the Fatou and the Julia set are completely invariant under R. Thus, any orbit remains in either one of the two sets and we can say that no orbit first admits nice behavior but becomes more complex later on. To get an overall picture of the dynamics of R, we can therefore simply study the Fatou and the Julia set.

Proposition 2.5. *The Fatou set is completely invariant. Consequently, the Julia set is, too.*

Proof. Obviously, if $(R^n)_{n\geq 0}$ is normal, then so is $(R^n)_{n\geq 1}$. This shows $R^{-1}(F) \subset F$ and the inclusion $R(J) \subset J$ follows. Similarly, if $(R^n)_{n\geq 0}$ is normal on $U \subset F$, then $(R^n)_{n\geq 1}$ is normal on $R(U)$. The latter is well-defined because $R(U)$ is again open as holomorphic maps are open. Adding a single map to a family of normal maps does not potentially annihilate normality. Hence, $(R^n)_{n\geq 0}$ is normal on $R(U)$, as well. We have proved $R(F) \subset F$ and as above, this inclusion implies $R^{-1}(J) \subset J$. In fact, since by surjectivity

$$R(F) \cup R(J) = \overline{\mathbb{C}} = R^{-1}(F) \cup R^{-1}(J)$$

we must have equalities everywhere. □

We already noted that the Fatou set contains attractive periodic orbits. Likewise, we can immediately classify some of the elements in the Julia set.

Proposition 2.6. *If p is a repelling periodic point for R, then p lies in the Julia set.*

Proof. Suppose for contradiction $(R^n)_{n\geq 0}$ is normal in a neighborhood of p. If q is the period of p, then $(R^{qn})_{n\geq 0}$ is normal near p, as well. By definition, there is a holomorphic function f defined on some compact neighborhood of p to which $(R^{qn})_{n\geq 0}$ converges on a subsequence $(n_k)_{k\geq 0}$. A contradiction follows immediately:

$$\infty > |f'(p)| = |\lim_{k\to\infty}(R^{qn_k})'(p)| = \lim_{k\to\infty}|R'(p)|^{qn_k} = \infty$$

□

We can ask ourselves what other points there might be in the Julia set. One of the main discoveries of early complex dynamics is that the repelling periodic points constitute the majority.

Theorem 2.7. *The Julia set is the closure of the repelling periodic points.*

Note that we have now characterized the Julia set independently of the Fatou set. We postpone the proof of this result for a while. First, we need to develop a deeper understanding of the Julia set. Since we are dealing with normality, recall Montel's theorem, which provides a considerably weak criterion for normality. For details on this, see for instance [6, p. 300].

Theorem 2.8 (Montel)**.** *Suppose $(f_n)_{n\geq 0}$ is a family of holomorphic maps. If there are three distinct points that are omitted by each f_n, then the family is normal.*

Consequently, this theorem tells us that on a neighborhood of the Julia set the family $(R^n)_{n\geq 0}$ omits at most two points. To be more precise, if U is an open set intersecting J, then

$$E_U = \overline{\mathbb{C}} \setminus \bigcup_{n\geq 0} R^n(U)$$

has cardinality at most 2. Moreover, if $V \subset U$ is any smaller open set still intersecting J, then $E_U \subset E_V$. Hence, for any point $z \in J$, we can pick an open neighborhood U so small that $E_z = E_U$ is independent of the choice of U. The set E_z is called the set of *exceptional points for z*. We will see that it is independent of the point $z \in J$ and that it will give us some useful information on the dynamics of R. Namely, the cardinality of $E_z \in \{0, 1, 2\}$ will determine a conjugacy of R to a simpler map.

Proposition 2.9. *Let $z \in J$. Firstly, the set E_z is independent of z and a subset of the Fatou set. Secondly, if E_z consists of exactly one point, then R is conjugate to a polynomial. If E_z consists of two points, then R is conjugate to either $z \to z^d$ or $z \to z^{-d}$.*

Proof. We first show the conjugacy statements and deduce afterwards the independence of z and the inclusion in the Fatou set. Assume E_z is non-empty. Pick a small open neighborhood U of z such that $E_z = E_U$.

$$R^{-1}(E_U) \subset \overline{\mathbb{C}} \setminus \bigcup_{n\geq -1} R^n(U) \subset E_U$$

shows that E_z is backward invariant. By surjectivity, $R^{-1}(E_z)$ contains as many elements as E_z. Hence, E_z either consists of a single fixed point, two fixed points, or an orbit of period two. In the first case, take a Möbius transformation ϕ that maps the point in E_z to ∞. Then the holomorphic map $\phi \circ R \circ \phi^{-1}$ has ∞ as a fixed point and no other point gets mapped to ∞. Put differently, this map has no pole besides ∞, meaning it is a polynomial. This proves the first case.
The second case is proved in a similar fashion. Pick a Möbius transformation ϕ that maps one point in E_z to 0 and the other one to ∞. Then $\phi \circ R \circ \phi^{-1}$ either has two fixed points at 0 and ∞, or maps 0 to ∞ and ∞ to 0. In both cases no other point gets mapped to 0 or ∞. In the first case this map is again a polynomial. But this time we know in addition that its only root is 0, which is therefore of multiplicity d. Hence, $\phi \circ R \circ \phi^{-1}$ is of the form $z \to Cz^d$. Of course, the constant C does not affect the conjugacy

class. For the second case, let ψ denote the chart $\psi(z) = \frac{1}{z}$. Then the composition $\psi \circ \phi \circ R \circ \phi^{-1}$ has fixed points 0 and ∞. By the first argument, this map is of the form $z \to Cz^d$, and so $\phi \circ R \circ \phi^{-1}$ is of the form $\psi^{-1}(Cz^d) = \frac{1}{Cz^d}$.
We have discussed the two different conjugacy classes R can have. In all cases above, the Möbius transformation depended a priori on E_z and hence on z. But a posteriori, we know that R is conjugate to one of the above maps and we can pick a conjugating map ϕ independent of any z. Then clearly, for any $z \in J$ the exceptional points are $\phi^{-1}(\infty)$ or $\phi^{-1}(\{0, \infty\})$, respectively. Moreover, this shows that there cannot be two points such that E_z is empty for one, but non-empty for the other. We can conclude that E_z is independent of z.
Lastly, we need to show that E_z is contained in the Fatou set. If E_z is empty, this is trivial. If R is conjugate to a polynomial with the singleton E_z corresponding to ∞, then E_z is a superattractive fixed point, hence in the Fatou set. If E_z has two points, R is conjugate to z^d or z^{-d} with the points in E_z corresponding to the origin and ∞. Either way, the points in E_z are both superattractive fixed points or a superattractive orbit of period two. □

Now that we know E_z does not depend on z, we drop the subscript and simply call it the set of *exceptional points* $E(R)$ or E. As a consequence, we can give a class of examples when the Fatou set is empty.

Corollary 2.10. *If the Julia set has non-empty interior, then it is all of $\overline{\mathbb{C}}$.*

Proof. With $U = \text{interior}(J)$, invariance of J yields

$$\overline{\mathbb{C}} \setminus E = \bigcup_{n \geq 0} R^n(U) \subset J$$

Since J is closed and E consists of at most two points, the assertion follows. □

Proposition 2.9 has more interesting consequences. We do not know yet how to efficiently compute the Julia set. However, we can show that it suffices to find a single point in it and compute its backward orbit. Note that, depending on the degree of R, this might still not be an easy task for a computer. It is convenient for maps of small degree, though.

Corollary 2.11. *For any $z \in J$, the backward orbit of z is dense in J.*

Proof. By invariance of J, the inclusion $\overline{\text{Orb}^-(z)} \subset J$ is trivial. Conversely, suppose $w \in J$. As a point in J, z surely is not an exceptional point by the last proposition. Hence, for any small neighborhood U of w we have $z \in \bigcup_{n \geq 0} R^n(U)$. But this means exactly that the backward orbit of z enters an arbitrary small neighborhood of w. □

Remark 2.12. *The proof of the last corollary actually shows the stronger statement that $J \subset \overline{\text{Orb}^-(z)}$ for any $z \in \overline{\mathbb{C}} \setminus E$.*

Thus, any backward orbit traverses densely through the Julia set. We can use this to see that there can be no proper, non-empty, closed, invariant subsets of J, since for such a subset K we would have

$$J = \overline{\text{Orb}^-(z)} \subset K$$

for any $z \in K$. In particular, whenever an attractive periodic point exists, we get yet another description of the Julia set.

Corollary 2.13. *If p is an attractive periodic point, then $J = \partial W^s(p)$.*

Proof. Consider some open neighborhood U of a point $z \in \partial W^s(p)$. By definition of $W^s(p)$, any orbit starting in $U \cap W^s(p)$ converges to p, while any orbit starting in $U \cap (\overline{\mathbb{C}} \setminus W^s(p))$ cannot enter $W^s(p)$. Hence, $(R^n)_{n\geq 0}$ cannot be equicontinuous on U and therefore $z \in J$. Thus, $\partial W^s(p)$ is a closed and invariant subset of the Julia set. Moreover, it cannot be empty, since this would mean that the Fatou set is the entire sphere, contradicting non-emptiness of J. $\square$

Remark 2.14. *Obviously, a more general result holds as well: If U is any completely invariant component of the Fatou set, then $J = \partial U$.*

The exceptional points have one more very important consequence.

Proposition 2.15. *The Julia set is perfect.*

Proof. We claim that for any point $z \in J$ we can find a second point $w \in J$ such that z is in the forward orbit of w but w is not in the forward orbit of z. Suppose for now the claim holds. Let U be an arbitrary small neighborhood of z and pick w as in the claim. $w \in J$ cannot be an exceptional point, hence it is contained in $\bigcup_{n\geq 0} R^n(U)$. Take $n \geq 0$ and $y \in U$ with $R^n(y) = w$. Since by the claim w is not in the forward orbit of z we must have $y \neq z$. Moreover, by invariance of the Julia set $y \in J \cap U$. This proves that z is an accumulation point, since U was arbitrarily small.
It remains to show the claim. If z is not a periodic point, then we can pick any $w \in \mathrm{Orb}^-(z)$. Suppose now that $R^n(z) = z$ and that no other point gets mapped to z under R^n (if there is another point, then we simply take that one). Take a Möbius transformation ϕ with $\phi(z) = \infty$. Then $\phi \circ R^n \circ \phi^{-1}$ has a fixed point at ∞ and maps no other point to ∞. As previously, $\phi \circ R^n \circ \phi^{-1}$ must be a polynomial. But then $\infty \in F(\phi \circ R^n \circ \phi^{-1})$ and so $z \in F(\phi \circ R^n) = F(R^n) \subset F(R)$, a contradiction. Therefore, there must be another point w with $R^n(w) = z$. By invariance, $w \in J$ and w is not in the forward orbit of z because $R^m(z) = w$ would imply $R^{n+m}(z) = z$, i.e. m is a multiple of n and $R^m(z) = z \neq w$. $\square$

We want to prove that the Julia set is exactly the closure of the repelling periodic points. That it is perfect was a key step for that. However, it still does not follow immediately. We first prove a slightly weaker statement.

Proposition 2.16. *The Julia set is contained in the closure of all the periodic points.*

Even to prove this weaker version we need one more auxiliary result. At first, this result may seem unrelated but note that critical values play a crucial role when picking inverse branches of functions. As a reminder: Critical points of R are points where the derivative of R vanishes. A critical value is the image of a critical point.

Proposition 2.17. *The number of critical points of R is at most $2d - 2$. If R is a polynomial, then the number of critical points is at most d.*

To prove this we want to use the Riemann-Hurwitz formula (see [13, p. 301]). Let us first review some results from the theory of covering maps. For details, the reader may consult [11].

Lemma 2.18. *$R\colon \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ is a branched covering map of degree d.*

Proof. We need to show that R is an unbranched covering map except on a nowhere dense set. Consider $R\colon \overline{\mathbb{C}} \setminus C \to \overline{\mathbb{C}} \setminus V$, where C denotes the set of critical points of R and V the set of critical values. By the inverse function theorem, R is locally injective on $\overline{\mathbb{C}} \setminus C$. As an open map, R is therefore a local homeomorphism on this set. Moreover, R is discrete, which is a consequence of the identity principle and compactness of $\overline{\mathbb{C}}$. Another consequence of compactness is that R is a proper map. Now let $z \in \overline{\mathbb{C}} \setminus V$ and $R^{-1}(z) = \{z_1, \ldots, z_k\}$. As a local homeomorphism, R admits neighborhoods U_j of z and W_j of z_j such that $R\colon W_j \to U_j$ is a homeomorphism. Since $\{z_1, \ldots, z_k\}$ is discrete, we can assume that the neighborhoods W_j are pairwise disjoint. Denote $W = \bigcup_{1\leq j\leq k} W_j$ and set $A = R((\overline{\mathbb{C}} \setminus C) \setminus W)$. The set

A is closed in $\overline{\mathbb{C}} \setminus V$ since R is proper and hence, the set $U = \bigcup_{1 \le j \le k} U_j \setminus R(A)$ is an open neighborhood of z. Finally, define $V_j = W_j \cap R^{-1}(U)$ to see that $R \colon \overline{\mathbb{C}} \setminus C \to \overline{\mathbb{C}} \setminus V$ is an unbranched covering map. By the Fundamental Theorem of Algebra, its degree has to be d. We use the identity principle together with compactness once more to see that the set C is finite. Hence, $R \colon \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ is a branched covering map of degree d. □

As a covering map, $R \colon \overline{\mathbb{C}} \setminus C \to \overline{\mathbb{C}} \setminus V$ gives rise to liftings of continuous maps. Using that R is holomorphic, the liftings of holormorphic maps will be holomorphic themselves.

Lemma 2.19. *If $f \colon \overline{\mathbb{C}} \setminus C \to \overline{\mathbb{C}} \setminus V$ is holomorphic and g is a lifting of f with respect to R (restricted to $\overline{\mathbb{C}} \setminus C$), then g is itself holomorphic.*

Proof. As a local homeomorphism, R is locally univalent and therefore locally biholomorphic. Hence, locally g is just the composition of a holomorphic inverse of R with f. □

This lemma conveys a particularly useful result when taking f to be the identity.

Corollary 2.20. *For any simply connected, pathwise connected and locally pathwise connected subset $D \subset \overline{\mathbb{C}} \setminus V$, any $z \in D$ and any $w \in R^{-1}(z)$, there exists a unique holomorphic inverse to R on D mapping z to w.*

Proof. The topological properties of D ensure the existence and uniqueness of a lifting with respect to the identity, which maps z to w (see [11, p. 26]). Now apply the last lemma. □

Remark 2.21. *Note that a simply connected open subset always satisfies pathwise connectivity and local pathwise connectivity.*

For the branch points of $R \colon \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ we can define their ramification index. The *ramification index* of a point p is the unique integer e_p for which there exists an open neighborhood U of p such that any point $z \in R(U) \setminus \{R(p)\}$ has exactly e_p preimages in U. In our case, e_p is exactly the multiplicity of $R'(z) = 0$ at p plus 1, where we consider the multiplicity to be 0 if $R'(p) \neq 0$. Hence, $e_p > 1$ if and only if p is a critical point.
Let us now prove the bounds on the number of critical points.

Proof of proposition 2.17. The Euler characteristic of the sphere is 2 ([25, p. 190]). The Riemann-Hurwitz formula states

$$\chi(\overline{\mathbb{C}}) = d \cdot \chi(\overline{\mathbb{C}}) - \sum_{p \in \overline{\mathbb{C}}} (e_p - 1)$$

which can be rewritten as

$$2d - 2 = \sum_{p \in \overline{\mathbb{C}}} (e_p - 1)$$

The summand $e_p - 1$ is 0 if p is not a critical point. On the other hand, if p is a critical point, then the summand is at least 1. Hence,

$$2d - 2 = \sum_{p \in \overline{\mathbb{C}}} (e_p - 1) \geq \#\{\text{critical points of } R\}$$

If R is a polynomial, then the ramification index of ∞ is d. The assertion follows from

$$d - 1 = \sum_{p \in \overline{\mathbb{C}} \setminus \{\infty\}} (e_p - 1) \geq \#\{\text{critical points of } R \text{ other than } \infty\}$$

□

Now we can prove that the periodic points are dense in the Julia set.

Proof of proposition 2.16. Proving a density statement for J can be reduced to proving a density statement for

$$K = J \setminus (\{\text{critical values of } R\} \cup \{\text{poles of } R\} \cup \{\infty\})$$

because J is perfect and the subtracted set is finite. Suppose for contradiction we can find an open, simply connected set U intersecting K that contains no periodic point. Again, we may assume without loss of generality that U does not contain any critical values, poles or ∞. By corollary 2.20, we can pick two different inverse branches S_1 and S_2 of R on U. Define a new family of functions on U by

$$g_n(z) = \frac{R^n(z) - S_1(z)}{R^n(z) - S_2(z)} \cdot \frac{z - S_2(z)}{z - S_1(z)}$$

None of the maps g_n can take value 0 or ∞ as we excluded the periodic points, the poles and ∞ itself from U. Lastly, we compute that each g_n cannot take value 1, either. Indeed, the following equations are equivalent:

$$\begin{gathered} g_n(z) = 1 \\ R^n(z)(S_2(z) - S_1(z)) = z(S_2(z) - S_1(z)) \\ R^n(z) = z \end{gathered}$$

where we used that S_1 and S_2 were chosen to be different branches. By Montel's theorem, the family $(g_n)_{n\geq 0}$ is normal. But then so is the family

$$R^n(z) = S_2(z) + \frac{z - S_2(z)}{z - S_1(z)} \cdot \frac{S_2(z) - S_1(z)}{g_n(z) - \frac{z - S_2(z)}{z - S_1(z)}}$$

However, U intersects the Julia set, a contradiction. □

We already know that the attractive orbits lie in the Fatou set. But we do not yet know where the neutral periodic points are located. However, we soon show that there are only finitely many. Since we only want to prove density of the repelling points, this suffices to conclude theorem 2.7 (here we again use the fact that the Julia set is perfect). Finiteness of the non-repelling periodic points is shown in two steps. First we prove that any attractive basin must contain a critical point and use proposition 2.17 to bound the number of attracting orbits. Then we perturb a given map to turn neutral orbits into attractive ones to which we can apply the previous bound.

Proposition 2.22. *For any attractive periodic point p there is a critical point in its immediate attractive basin.*

Proof. Suppose first p is a fixed point. Assume for contradiction there is no critical point in $A(p)$. In particular, there is no critical value of R, and hence of R^n, either. By corollary 2.20, for any open simply connected neighborhood $U \subset A(p)$ of p, there is an inverse branch S_n of R^n on U with $S_n(p) = p$. As the images of S_n are in $A(p)$, the family $(S_n)_{n\geq 0}$ is normal on U. A contradiction arises as in the proof of proposition 2.6. Namely, if $(S_{n_k})_{k\geq 1}$ is some convergent subsequence with limit function S, then

$$\infty > |S'(p)| = \lim_{k\to\infty} |(S_{n_k})'(p)| = \lim_{k\to\infty} |\frac{1}{(R^{n_k})'(p)}| = \infty$$

Suppose now p is periodic of period $n > 1$. By the previous part $A(p, R^n)$ contains a critical point of R^n. Since $(R^n)'(p) = \prod_{0\leq j\leq n-1} R'(R^j(p))$, R must have a critical point in the immediate attractive basin of $\mathrm{Orb}(p)$. □

Corollary 2.23. *The number of attracting periodic orbits is at most $2d-2$. If R is a polynomial, there are at most $d-1$ besides ∞.*

We can use these bounds to also count neutral orbits by perturbing the rational map to one that has attractive orbits where R had neutral ones. However, with our technique not every neutral orbit can be made attractive and so the resulting bound is not sharp. We sketch its proof so that later it can be compared to a proof of a sharper bound of $d-1$ for polynomials using polynomial-like maps. With the use of quasi-conformal maps, Shishikura managed to turn every neutral orbit into an attractive one and obtained the sharp bound of $2d-2$. The bound for polynomials, which we show later, is therefore sharp.

Proposition 2.24. *The number of attracting periodic orbits plus half the number of neutral periodic orbits is at most $2d-2$.*

Sketch of proof. Consider N neutral periodic orbits of R. Define $S(z,w) = (1-w)R(z) + w$. We can pick a branch $L = \{re^{i\theta} \mid r > 0\}$ such that for all $w \in L$ close to 0 the map $S(\cdot, w)$ has $N/2$ attractive periodic orbits close to the neutral orbits of R (for details see [3, p. 111–112]). Since $S(\cdot, w)$ is a small analytic perturbation of R, if w is close enough to 0, all attractive orbits of R remain attractive orbits for $S(\cdot, w)$. It follows from the last corollary applied to $S(\cdot, w)$ that the number of attractive orbits of R plus $N/2$ is at most $2d-2$ (in particular, R cannot have infinitely many neutral orbits). □

We have successfully bounded the number of non-repelling periodic orbits. Theorem 2.7 now follows from this fact together with proposition 2.15 and 2.16.
Back at the beginning, we described the Julia set as the region with exotic behavior. Both the density of the repelling orbits and the property that the backward orbit of any point in the Julia set is dense can be seen as an explanation. The latter has sort of an equivalent property in the forward direction.

Proposition 2.25. *For any open set U intersecting J, there exists an $N \geq 0$ such that $J = R^N(U \cap J)$.*

Proof. Let z be a repelling periodic point in $U \cap J$ of period n. Take a small neighborhood $z \in V \subset U$ such that $V \subset R^n(V)$. This way, $V \subset R^n(V) \subset R^{2n}(V) \subset \dots$ is an increasing sequence. If V is small enough, then it does not contain an exceptional point. Hence,

$$\bigcup_{k \geq 0} R^{kn}(V) = \overline{\mathbb{C}} \setminus E \supset J$$

By compactness, $J \subset R^N(V) \subset R^N(U)$ for some large N and by invariance $J = R^N(U \cap J)$. □

We have obtained quite a few intricate results on the Julia set. Let us now discuss some properties of the Fatou set. We first review a few topological properties and then verify with Sullivan's theorem that the dynamics on the Fatou set really are simple.

Proposition 2.26. *The Fatou set has at most two connected, simply connected and completely invariant subsets.*

Proof. Recall from the proof of proposition 2.17 that

$$2d - 2 = \sum_{p \in \overline{\mathbb{C}}} (e_p - 1)$$

where e_p is the ramification index at p. If U is a component as described, then the restriction of R to U is a branched covering map $U \to U$, with branch points the critical points in U. By the Riemann-Hurwitz formula

$$\chi(U) = d \cdot \chi(U) - \sum_{p \in U} (e_p - 1)$$

and since U is homeomorphic to a disk, we have $\chi(U) = 1$. Therefore,

$$d - 1 = \sum_{p \in U} (e_p - 1)$$

If $U_1, \ldots, U_n$ all are such components, then by the first formula

$$2d - 2 = \sum_{p \in \overline{\mathbb{C}}} (e_p - 1) \geq \sum_{1 \leq j \leq n} \sum_{p \in U_j} (e_p - 1) = n \cdot (d - 1)$$

□

In fact, we cannot only count the number of such specific components, but all of them. Recall some basic terminology: Just like we call a point periodic if $R^n(z) = z$ for some n, we say a component of the Fatou set is periodic if $R^n(U) = U$ for some n. The notions of eventually periodic and preperiodic (i.e. eventually periodic but not periodic) are defined analogously.

Proposition 2.27. *The number of components of the Fatou set can only be 0, 1, 2 or* ∞*.*

Proof. Assume that there are only finitely many components. Let U be any one of them. By invariance, each $R^n(U)$ is also some component of the Fatou set. Since there are only finitely many, U must be periodic. Furthermore, by the finiteness assumption there is some integer N such that every component is periodic with the same period N. In particular, they are completely invariant for $S = R^N$. By the last proposition, at most two of them can be simply connected. Suppose for contradiction that there are more than two. Then we can pick a component U that is not simply connected. By conjugating S with a Möbius transformation, we may assume that ∞ is in a different component of the Fatou set than U. Now take a loop γ in U that is not nullhomotopic. Since U is invariant, $(S^n)_{n \geq 0}$ is bounded on γ. If D denotes the component of $\overline{\mathbb{C}} \setminus \gamma$ not containing ∞, then $(S^n)_{n \geq 0}$ is bounded on D by Cauchy's integral formula. Hence, $(S^n)_{n \geq 0}$ is normal on D. But by hypothesis $D \cap J$ is non-empty. This is a contradiction because the Fatou set of R and S coincide. The latter can be seen as follows: Clearly, if $(R^n)_{n \geq 0}$ is normal, then so is $(R^{Nn})_{n \geq 0}$. Conversely, if $(R^{Nn})_{n \geq 0}$ is equicontinuous, then so is each $(R \circ R^n)_{n \geq 0}$ and a finite union of equicontinuous families is again equicontinuous. □

Dealing with infinitely many components still is a hard task. Luckily, Sullivan proved that there is no need for that. He showed that there are finitely many periodic components such that any other component is a preimage of one of the periodic ones. Since in the study of dynamics we are not interested in the first finitely many iterates of an orbit, it therefore suffices to study the dynamics on the periodic components.

Theorem 2.28 (Sullivan, part 1)**.** *Every component of the Fatou set is eventually periodic. Moreover, there are only finitely many periodic components.*

For the first statement, see [28, p. 3] or [5, p. 71], and for the second [18, p. 6]. For the periodic components Sullivan gave a complete classification of all possible dynamics. He proved that the following scenarios are all that can happen.

Definition 2.29. *A periodic component of the Fatou set is called Sullivan domain. Furthermore, a Sullivan domain is*

1. *an attractive domain if it is the immediate attractive basin of an attractive, but not superattractive periodic point.*
2. *a superattractive domain if it is the immediate attractive basin of a superattractive periodic point.*
3. *a parabolic domain if its boundary contains a periodic point p with eigenvalue 1, whose period divides the one of the domain itself, and the forward orbit of every point in the domain converges to the orbit of p.*

4. *a Siegel disk if it is simply connected and on this domain R is analytically conjugate to a rotation.*
5. *a Herman ring if it is conformally equivalent to an annulus and on this domain R is analytically conjugate to a rotation.*

Theorem 2.30 (Sullivan, part 2)**.** *Every Sullivan domain is of one of the above types. Moreover, attracting and parabolic domains both contain infinite forward orbits of critical points and the boundaries of rotation domains are contained in the closure of the forward orbits of the critical points.*

The original discussion is in [18, p. 404] and another proof can be found in the textbook [5, p. 74-79]. The additional result in this theorem enables us to study the behavior of the orbits of critical points to deduce properties of the Fatou and the Julia set. Let us give a few examples of how to do that.

Corollary 2.31. *If every critical point of R is preperiodic (i.e. eventually periodic but not periodic), then $J = \overline{\mathbb{C}}$.*

Proof. The Fatou set cannot have a superattractive Sullivan domain since such contains a periodic critical point. Any other Sullivan domain is excluded as well, since they all require the existence of infinite orbits of critical points. □

For polynomials, we can include an additional condition.

Corollary 2.32. *Suppose R is a polynomial with superattractive periodic points $p_1, \ldots, p_n$ and ∞ (where possibly $n = 0$). If every critical point is preperiodic or has unbounded orbit, then*

$$\overline{\mathbb{C}} = J \cup W^s(\infty) \cup \bigcup_{1 \le j \le n} W^s(p_j)$$

Proof. Similar to the proof of the last corollary, if all the critical orbits are preperiodic or enter $W^s(\infty)$, then there cannot be any attractive or parabolic domains, nor Siegel disks or Herman rings. Hence, the Fatou set can only consist of superattractive domains and its preimages. But for a superattractive periodic point p the attractive basin is completely invariant and there are no preimages aside from $W^s(p)$. □

Before, we noted that there can be at most two simply connected, completely invariant components. Conversely, if we are given such, then we can say a lot about the nature of the Fatou set.

Corollary 2.33. *If the Fatou set has two simply connected, completely invariant components, then these are the only ones and they are superattractive, attractive or parabolic Sullivan domains.*

Proof. Let U_1 and U_2 denote the two given components. From the proof of proposition 2.26 we know

$$2d - 2 = \sum_{p \in \overline{\mathbb{C}}} (e_p - 1) \ge \sum_{1 \le j \le 2} \sum_{p \in U_j} (e_p - 1) = 2d - 2$$

Thus, all the critical points are inside $U_1 \cup U_2$. By the second part of Sullivan's theorem, there cannot be any other Sullivan domains since they all require critical orbits inside of them or on their boundary. Moreover, by the first part of Sullivan's theorem the only components of the Fatou set are U_1, U_2 and their preimages. But by complete invariance U_1 and U_2 have no preimages other than themselves and it follows that they are the only components. Lastly, as we observed that all the critical points are inside U_1 and U_2, they cannot be Siegel disks or Herman rings. □

As another application, Sullivan's theorem can be used to prove a sufficient condition for expansivity on the Julia set.

Theorem 2.34. *If $J \cap \overline{\mathrm{Orb}^+(C)} = \emptyset$, where C denotes the set of critical points, then R is expanding on J, i.e.*

$$\forall K > 1\ \exists N \in \mathbb{N} \text{ such that } \forall n \geq N\ \forall z \in J \colon |(R^n)'(z)| > K$$

The converse is true as well: If R is expanding on J, then $J \cap \overline{\mathrm{Orb}^+(C)} = \emptyset$.

We slightly modify the proof from [3, p. 119].

Proof. Let D be a simply connected domain with $D \cap \overline{\mathrm{Orb}^+(C)} = \emptyset$ and $D \cap J \neq \emptyset$. By corollary 2.20, we can take inverse branches I_n of every R^n as the critical values of R^n are exactly $R^n(C) \subset \overline{\mathbb{C}} \setminus \overline{D}$. We first claim that the family $(I_n)_{n\geq 0}$ is normal in a neighborhood of $D \cap J$. To see this, note that a repelling fixed point of R^n is an attractive fixed point of I_n. Hence, if p is a repelling periodic point for R of period q, then there is a small neighborhood U_p of p on which the family $(I_q n)_{n\geq 1}$ is normal, and hence also the family $(I_n)_{n\geq 0}$. By theorem 2.7, these points p are dense in J. By compactness, we can cover J by finitely many such neighborhoods $U_{p_1}, \ldots, U_{p_k}$. It is immediate from the definition of normality that the family $(I_n)_{n\geq 0}$ is normal on the finite union $\bigcup_{1\leq j\leq k} U_{p_j} \supset J$. This concludes the claim.
Let U denote the neighborhood on which $(I_n)_{n\geq 0}$ is normal. Take any convergent subsequence of the family. We make a second claim: This subsequence necessarily converges to a constant function. Let us show how the first statement of the theorem follows from this claim and prove the claim afterwards. Since the subsequence converges to a constant function, the derivatives I_n' limit to zero on this subsequence, i.e.

$$\forall K > 1\ \exists N \in \mathbb{N} \text{ such that for infinitely many } n \geq N\ \forall z \in J \colon |I_n'(z)| < \frac{1}{K}$$

The convergent subsequence was arbitrary and by normality, any subsequence of $(I_n)_{n\geq 0}$ has a convergent subsequence. Hence, the inequality actually holds for all $n \geq N$ and not just for infinitely many n. The assertion now follows from $I_n' = \frac{1}{(R^n)'}$ as the maps I_n are inverse branches of R^n. It remains to prove the second claim.
First note that the Fatou set is non-empty by hypothesis. Corollary 2.10 tells us that the Julia set must have empty interior. Now let I denote any limit function of the family $(I_n)_{n\geq 0}$. We will show that the image of I is contained in the Julia set and, hence, has empty interior itself. If so, then I must be constant, because any non-constant holomorphic map is open. Suppose for contradiction I has an image point outside of J. This means that there exists a neighborhood V of J, a point $z \in U$, and a sequence $n_k \to \infty$ such that for each k the point $I_{n_k}(z)$ lies outside V. By invariance, the point z cannot be in the Julia set. Equivalently, we can consider the sequence of points $z_k = I_{n_k}(z)$ such that each $R^{n_k}(z_k) = z \in U \setminus J$. Let z_0 be an accumulation point of $(z_k)_{k\geq 1}$. Since each z_k was outside of V, z_0 lies in the Fatou set of R and consequently $(R^n)_{n\geq 0}$ is normal in a neighborhood of z_0. By uniform convergence, we have

$$\lim_{k\to\infty} R^{n_k}(z_0) = \lim_{k\to\infty} R^{n_k}(z_k) = z \in U \setminus J$$

But at the same time, the assumption $J \cap \overline{\mathrm{Orb}^+(C)} = \emptyset$ together with Sullivan's theorem assures that the Fatou set only has attractive or superattractive Sullivan domains. As z_0 is in the Fatou set, the sequence $R^{n_k}(z_0)$ must therefore converge to a (super-)attractive periodic point. Thus, z is such a point. Then by Sullivan's theorem again, we know that there must be a critical orbit converging to z, which contradicts $U \cap \overline{\mathrm{Orb}^+(C)} = \emptyset$. This proves the second claim.
Next, we want to show the converse statement. Suppose R is expanding. The definition of expansivity rules out the existence of critical points inside J. Thus, the Fatou set cannot have Siegel disks or Herman rings. If there was a parabolic domain, then the Julia set would contain a parabolic periodic point p. But then $|R^{nq}(p)| = 1$, where q denotes the period of p, contradicting expansivity. Hence, the Fatou

set consists only of attractive and superattractive domains as well as their preimages. In any case, the critical orbits converge to the attracting orbits, proving $J \cap \overline{\mathrm{Orb}^+(C)} = \emptyset$. □

To conclude the discussion of rational dynamics, we strengthen proposition 2.25. Namely, under the right hypothesis, not only can we map a small piece of the Julia set onto itself via R^N, but we can map into it with a quasi-isometry.

Definition 2.35. *A function $f\colon X \to X$ on a metric space is a K-quasi-isometry if*

$$\frac{1}{K} d(x,y) \leq d(f(x), f(y)) \leq K d(x,y) \text{ for any } x, y \in X$$

A subset $A \subset X$ is quasi-self-similar if there exist $K > 0$ and $r_0 > 0$ such that for any $r < r_0$ and any $x \in A$, there is a K-quasi-isometry f such that $f(\{\frac{1}{r}z \mid z \in D_r(x) \cap A\}) \subset A$.

The definition means that a small piece in A can be expanded to full size (via multiplication with $\frac{1}{r}$) and then mapped into A by a quasi-isometry. Now we consider A to be the Julia set of a rational map. As a reference for the next result, see [27, p. 48].

Theorem 2.36. *If $J \cap \overline{\mathrm{Orb}^+(C)} = \emptyset$, where C again denotes the set of critical points, then J is quasi-self-similar.*

In the next chapter we will investigate the special case where R is a polynomial. One of the main results will again be linked to the behavior of the critical orbits.

2.2 Polynomial Dynamics

Let $R = p\colon \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be a polynomial of degree $d \geq 2$. Since ∞ always is a superattractive fixed point for p, whenever we speak of a (super-)attractive periodic orbit we implicitly mean a *finite* such orbit, i.e. exclude ∞.
Towards the end of the last chapter we studied what types of Sullivan domains can occur and we want to begin this chapter by continuing this discussion for polynomials. For these, one type is automatically excluded.

Proposition 2.37. *The Sullivan domains of a polynomial are never Herman rings.*

Proof. Suppose there was a Herman ring H. Unraveling the definition yields a function ϕ mapping H to an annulus such that $\phi \circ p \circ \phi^{-1}$ is a rotation on that annulus. Pick any circle inside the annulus. Then the image of that circle under ϕ^{-1} is a p-invariant closed Jordan curve γ inside H. Let U denote the connected component of $\overline{\mathbb{C}} \setminus \gamma$ not containing ∞. Then U is open and simply connected. By the maximum principle, for any $n \geq 0$:

$$\sup_{z \in \overline{U}} |p^n(z)| = \sup_{z \in \partial U} |p^n(z)| = \sup_{z \in \gamma} |p^n(z)| \leq \sup_{z \in \gamma} |z| < \infty$$

Hence, each map p^n is uniformly bounded on U. But since $U \cap J \neq \emptyset$, it follows from Montel's theorem that $\bigcup_{n \geq 0} p^n(U)$ leaves out at most two points, a contradiction. □

On the other hand, we already noted that we do not rely on the study of the Sullivan domains themselves but can instead investigate the behavior of the critical orbits. For polynomials, this approach proves even more useful. Recall that the Julia set is exactly the boundary of $W^s(\infty)$ (see corollary 2.13). Now, we can give sufficient and necessary conditions for the Julia set to be connected. This theorem is one of the main results in polynomial dynamics. To make sense of the first condition recall that any non-constant holomorphic map (on any Riemann surface) can be conjugated locally to $z \to z^k$ for some $k \geq 1$ (see, for example, [11, p. 10]).

Theorem 2.38. *The following are equivalent:*

1. *The map conjugating p to $z \to z^d$ near ∞ extends to $W^s(\infty)$.*
2. *$W^s(\infty)$ is simply connected.*
3. *J is connected.*
4. *The orbit of any critical point is bounded.*

In order to prove it we need to develop a technique for extending the conjugation near ∞. Let ϕ be the map conjugating p with $z \to z^d$ in a neighborhood U of ∞. It satisfies the functional equation $\phi(p(z)) = \phi(z)^d$. An explicit formula[2] is given by

$$\phi(z) = \lim_{n\to\infty} (p^n(z))^{d^{-n}}$$

If we set $G(z) = \log|\phi(z)|$, then the functional equation for ϕ translates to $G(p(z)) = d \cdot G(z)$. This allows us to extend G to a map on all of $W^s(\infty)$ by setting $G(z) = \frac{1}{d^n} G(p^n(z))$, where n is so large that $p^n(z) \in U$. With $G(z) = 0$ on $K = \mathbb{C} \setminus W^s(\infty)$ it is now defined on all of $\overline{\mathbb{C}}$. It is the *Green's function*[3] of $W^s(\infty)$. ϕ is sometimes called the *Boettcher coordinate.*

Now take r large such that $\{z \mid G(z) > r\} \subset U$. Observing $p(\{G(z) > r\}) \subset \{G(z) > d \cdot r\}$, we can extend $\phi(z) = \phi(p(z))^{1/d}$ to $\{G(z) > r/d\}$ provided $\{G(z) > r\}$ contains no critical value of p (as the absence of critical values allows us to pick a well-defined root).
We are ready to prove the last theorem.

Proof of Theorem 2.38. $4 \Rightarrow 1$: As discussed in the preparation of this proof, we can extend the conjugacy ϕ as long as $\{z \mid G(z) > r/d\}$ contains no critical points. In particular, if the orbit of any critical point is bounded, we can extend ϕ to all of $W^s(\infty)$.
$1 \Rightarrow 2$: The image of ϕ is $\{|z| > 1\}$ and so, $W^s(\infty)$ is homeomorphic to a disk.
$2 \iff 3$: For a polynomial, $W^s(\infty)$ always is connected. Moreover, an open set is simply connected if and only if its boundary is connected, see [2, p. 343].
$3 \Rightarrow 4$: Suppose there is a critical point with unbounded orbit. Then ϕ can only be extended up to some level curve $\{G = r\}$ of Green's function that contains a critical point of p. Near p $\phi(z)$ takes different values depending on the direction from which z approaches p. The directions correspond to cusps formed by the level line $\{G = r\}$ (see figure 1). In particular, the set $\{G < r\}$ is encompassed by at least two closed Jordan curves. Each of these curves encircles a subset of J for if one did not, then this curve would enclose a subset of $W^s(\infty)$ on which G is harmonic, and by the maximum/minimum principle for harmonic functions G would be constant on this subset. But then G would be constant in all of $\{G > r\}$, a contradiction. Each Jordan curve encircling a subset of J contradicts connectedness of J. ☐

As depicted in figure 1, under a slightly stronger hypothesis we can say more about the disconnectedness of the Julia set in the part "$3 \Rightarrow 4$".

Theorem 2.39. *If the orbit of any critical point is unbounded, then J is totally disconnected.*

[2]Note that we can write

$$\lim_{n\to\infty} (p^n(z))^{d^{-n}} = z \cdot \prod_{n\geq 1} \left(\frac{p^n(z)}{p^{n-1}(z)^d}\right)^{d^{-n}}$$

and this infinite product converges because

$$\sum_{n\geq 1} d^{-n}(\log(p^n(z)) - \log(p^{n-1}(z)^d))$$

converges uniformly and absolutely.

[3]For an open set $U \subset \overline{\mathbb{C}}$, the Green's function is $G\colon \overline{\mathbb{C}} \to [0, \infty)$ such that G is harmonic in U, vanishes outside U, and $G(z) - \log(|z|)$ is bounded near ∞.

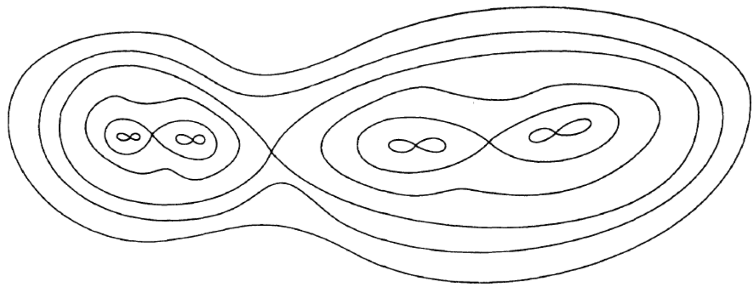

Figure 1: Level curves of Green's function with J totally disconnected, [5, p. 66].

Proof. By hypothesis we may take a small closed neighborhood $G \subset W^s(\infty)$ of ∞ such that $J \subset D = \overline{\mathbb{C}} \setminus G$ and $p(G) \subset \text{interior}(G)$. Pick N large so that any critical point gets mapped into the interior of G under p^N. Then, p^n has no critical point in $p^{-n}(\overline{D})$ for any $n \geq N$ and therefore no critical value in $\overline{D}$. In particular, by corollary 2.20, given any $z \in J$ there is an inverse map of p^n defined on $\overline{D}$ sending $p^n(z) \in J$ to z, denoted by i_n. As i_n maps $\overline{D}$ into $\overline{\mathbb{C}} \setminus G$, $(i_n)_{n \geq N}$ is a bounded, hence normal family.
It follows from remark 2.12 that for any $w \in D \cap F$ that is not an exceptional point, $i_n(w)$ has an accumulation point in J. As there are at most two exceptional points, there exists a limit function of $(i_n)_{n \geq N}$ that maps $D \cap F$ into J. To see this, we can consider a diagonal argument on a dense countable subset of $D \cap W^s(\infty)$ and use normality. Note that by invariance of J, all of D is mapped into J by a limit function.
By corollary 2.10, J has empty interior and hence, the above limit function must be constant as holomorphic functions are always open. Thus, the diameter of $i_n(\overline{D})$ tends to 0 as $n \to \infty$. Since ∂D is disjoint from J and J is completely invariant, each $i_n(\partial D)$ is also disjoint from J. Thus, $i_n(\partial D)$ separates the subset of J containing z and the subset of J that lies outside $i_n(\overline{D})$. That the diameter of $i_n(\overline{D})$ tends to 0 shows that $\{z\} = \bigcap_{n \geq 0} i_n(\overline{D})$ must be a connected component. □

It sometimes is convenient to study the so-called filled-in Julia set. It is usually denoted by K and is simply given by $\overline{\mathbb{C}} \setminus W^s(\infty)$, i.e. consists of the Julia set and all the finite components of the Fatou set.

Remark 2.40. *With an argument as in corollary 2.32, we see that under the hypothesis of the last theorem, the Fatou sets consists only of $W^s(\infty)$ (there cannot be any other (super-)attractive periodic points as all the critical orbits enter the basin of ∞). Hence, K and J coincide and the filled-in Julia set is totally disconnected, too.*

There is an alternative proof of the last theorem, which considers inverse images of annuli around ∞, found in [3, p. 124–126]. The alternative proof has the advantage that we get the following result on the dynamics on the Julia set for free.

Theorem 2.41. *If the orbit of any critical point is unbounded, then p admits chaotic behavior on its Julia set.*

Hence, if the critical orbits are unbounded, then the dynamics on the Julia set are as exotic as they can get.
Next, we will investigate a certain class of polynomials for which the theorems 2.38 and 2.39 convey a strong consequence. Let p_c denote the polynomial $z^d + c$. These polynomials are particularly important in the case $d = 2$. There, any polynomial can be conjugated by an affine map to one of the form $z^2 + c$. On the other hand, any two such p_{c_1} and p_{c_2} cannot be conjugated unless $c_1 = c_2$. Thus, the $\{p_c\}_{c \in \mathbb{C}}$ represent exactly the equivalence classes of polynomial dynamical systems of degree 2 under analytic conjugation (and also other notions of conjugacies of dynamical systems). Hence, it suffices to study

these systems in order to classify any possible dynamics in degree 2. For $d \geq 3$ this is not true. However, since most of the proofs do not differ for larger d, we usually consider any $d \geq 2$ for generality. The form $z^d + c$ has the following nice effect. The only critical points of p_c are 0 and ∞. With the theorems 2.38 and 2.39 we discover the following dichotomy.

Corollary 2.42. *The Julia set is either connected or totally disconnected. Connectedness corresponds to* $0 \notin W^s(\infty)$ *and total disconnectedness to* $0 \in W^s(\infty)$*.*

In fact, we are free to study either one of the Julia or the filled-in Julia set as the dichotomy transfers to K.

Proposition 2.43. *The filled-in Julia set K is connected if and only if the Julia set J is. The same is true for total disconnectedness.*

Proof. To prove the first statement, we use the following result, see for instance [6, p. 202]. An open connected subset of $\overline{\mathbb{C}}$ is simply connected if and only if its complement is connected. With theorem 2.38 we conclude that J is connected if and only if $W^s(\infty)$ is simply connected if and only if K is connected. Secondly, if J is totally disconnected, then by theorem 2.38 together with corollary 2.32, we see that the Fatou set has $W^s(\infty)$ as its only component, i.e. $K = J$. $\square$

This dichotomy motivates the definition of the Mandelbrot set.

3 The Mandelbrot Set $\mathcal{M}$

Let $p_c(z) = z^d + c$. By corollary 2.42 the Julia set of p_c either is connected or totally disconnected. We can ask ourselves which parameters c correspond to which case. To investigate this question we introduce the connectedness locus, commonly known under the name *Mandelbrot set*.

Definition 3.1. *The Mandelbrot set is*

$$\mathcal{M} = \{c \mid p_c \text{ has a connected Julia set}\}$$

If $d \geq 3$, it is usually called *Multibrot set*. However, we will always refer to it as the Mandelbrot set. As it turns out, $\mathcal{M}$ does not inherit a simple structure and the above question really is interesting.
By theorem 2.38 we can characterize $\mathcal{M}$ alternatively as the set $\{c \mid (p_c^n(0))_{n\geq 0} \text{ is bounded}\}$. Instead of requiring finiteness we can give a sharp bound for $(p_c^n(0))_{n\geq 0}$.

Proposition 3.2.

$$\mathcal{M} = \{c \mid \forall n \geq 0\colon\ |p_c^n(0)| \leq 2\}$$

Consequently, $\mathcal{M}$ is a closed subset of the disk of radius two.

Proof. We first show that if $c \in \mathcal{M}$, then $|c| \leq 2$. Indeed, if $|c| = 2 + \delta > 2$, then by induction $|p_c^n(0)| \geq 2 + 2^{n-1}\delta$:

$$|p_c^{n+1}(0)| \geq |p_c^n(0)|^d - |c| \geq (2 + 2^{n-1}\delta)^d - (2+\delta) \geq (2 + 2^{n-1}\delta)^2 - (2+\delta) \geq 2 + 2^n\delta$$

Hence, $(p_c^n(0))_{n\geq 0}$ is unbounded and $c \notin \mathcal{M}$. Next, suppose $c \in \mathcal{M}$ but $|p_c^n(0)| = 2 + \delta > 2$ for some $n \geq 1$. Using $|c| \leq 2$ we can prove by induction $|p_c^{n+k}(0)| \geq 2 + 4^k\delta$:

$$|p_c^{n+k+1}(0)| \geq |p_c^{n+k}(0)|^d - |c| \geq (2 + 4^k\delta)^d - |c| \geq (2 + 4^k\delta)^2 - 2 \geq 2 + 4^{k+1}\delta$$

This contradicts $c \in \mathcal{M}$. □

This allows to write somewhat efficient algorithms for computing images of $\mathcal{M}$. Namely, for a fine-mesh lattice of sample parameter points c in the disk of radius two one can compute the first, say 1000, iterates $p_c^n(0)$ and check whether they exceed the value two. But this only yields a rough approximation. Since an analytical approach to when this bound holds is difficult we need different tools to study $\mathcal{M}$. One of the topological properties usually studied first is connectedness of a set. An easy result can be deduced using the maximum principle from complex analysis.

Corollary 3.3. *The connected components of the interior of $\mathcal{M}$ are simply connected.*

Proof. Take a closed Jordan curve γ in one of the components. Let U denote the connected component of $\overline{\mathbb{C}} \setminus \gamma$ not containing ∞. Then U is open and simply connected. Note that for a fixed n, the map $p_c^n(0)$ is a polynomial expression in c and hence, the map $c \to p_c^n(0)$ is holomorphic. By the maximum principle and the last proposition, we have for any $n \geq 1$:

$$\sup_{c\in\overline{U}} |p_c^n(0)| = \sup_{c\in\partial U} |p_c^n(0)| = \sup_{c\in\gamma} |p_c^n(0)| \leq 2$$

This shows that $\overline{U}$ lies in the Mandelbrot set. In particular, it is part of the connected component in which we picked γ. Therefore, we can contract γ along U, proving that it must be nullhomotopic. □

While the interior of $\mathcal{M}$ has infinitely many connected components, which we will become clear later in theorem 3.5, we get a connected set if we add the boundary. To prove connectedness of $\mathcal{M}$, we continue the discussion of Green's function of $W^s(\infty)$ for the polynomials p_c.
Let ϕ_c be the map conjugating p_c with $z \to z^d$ in a neighborhood near ∞ and set $G_c(z) = \log|\phi_c(z)|$. As in the last chapter, extend G_c to a map on all of $\overline{\mathbb{C}}$ and extend ϕ_c as long as $\{z \mid G(z) > r\}$ contains no critical values of p_c. In this special case, 0 is the only finite critical point. Thus we can extend ϕ_c to the set $\{z \mid G_c(z) > G_c(0)\}$. Note that $G_c(z)$ depends continuously on c for any fixed z. Hence, $G_c(c) \to 0$ as $c \to \mathcal{M}$ because if $c \in \mathcal{M}$, then $(p_c^n(c))_{n\geq 0}$ is bounded and

$$G_c(z) = \lim_{n\to 0} \frac{1}{d^n} \log(|p_c^n(z)|) = 0$$

If $G_c(0) > 0$ (or, equivalently $c \notin \mathcal{M}$), then

$$G_c(c) = G_c(p_c(0)) = d \cdot G_c(0) > G_c(0)$$

tells us that ϕ_c is defined at the point c. Hence,

$$\Phi\colon c \to \phi_c(c) = \lim_{n\to\infty} (p_c^n(c))^{d^{-n}}$$

is a well-defined holomorphic map on $\overline{\mathbb{C}} \setminus \mathcal{M}$. From the observation $G_c(c) \to 0$ as $c \to \mathcal{M}$ it follows that Φ is proper as a map into $\overline{\mathbb{C}} \setminus \overline{D}$, where D denotes the unit disk. As a proper, hence closed holomorphic map, Φ maps $\overline{\mathbb{C}} \setminus \mathcal{M}$ surjectively onto $\overline{\mathbb{C}} \setminus \overline{D}$. We can use the argument principle

$$\#\Phi^{-1}(w) = \frac{1}{2\pi i}\int_\Gamma \frac{\Phi'(z)}{\Phi(z) - w} dz$$

where Γ is a path encompassing $\Phi^{-1}(w)$, to see that Φ also is injective. Indeed, the right hand side is continuous in w and the left hand side takes values in the natural numbers. Therefore, the right hand side is locally constant, hence constant $\equiv \Phi^{-1}(\infty) = 1$ on the connected set $\overline{\mathbb{C}} \setminus \overline{D}$. It follows that

$$\Phi\colon \overline{\mathbb{C}} \setminus \mathcal{M} \to \overline{\mathbb{C}} \setminus \overline{D}$$

is a holomorphic diffeomorphism. We have proved the following result.

Theorem 3.4. *$\mathcal{M}$ is connected and full (i.e. $\overline{\mathbb{C}} \setminus \mathcal{M}$ is connected).*

Proof. We showed above that $\overline{\mathbb{C}}\setminus\mathcal{M}$ is diffeomorphic to $\overline{\mathbb{C}}\setminus\overline{D}$, hence it is connected and simply connected. Moreover, an open connected subset of $\overline{\mathbb{C}}$ is simply connected if and only if its complement is connected. □

Now that we established connectedness, we focus our attention on the components of the interior of $\mathcal{M}$. Surprisingly, many of those are characterized by a single property.

Theorem 3.5. *If p_{c_0} has a finite attracting periodic orbit of length m, then c_0 belongs to the interior of $\mathcal{M}$. Moreover, the connected component of the interior of $\mathcal{M}$ around c contains only points c' for which $p_{c'}$ has an attractive periodic orbit of the same length m. The orbit depends analytically on c'.*

Proof. Let $z(c_0)$ be an attracting periodic point for p_{c_0} of period m. Consider the map

$$Q\colon \mathbb{C} \times \mathbb{C} \to \mathbb{C},\ Q(c,z) = p_c^m(z) - z$$

It satisfies $Q(c_0, z(c_0)) = 0$ and as a polynomial expression in both z and c, it is holomorphic. Since

$$(\frac{\partial}{\partial z}Q)(c_0, z(c_0)) = \prod_{0\leq j\leq m-1} p'_{c_0}(p^j_{c_0}(z(c_0))) - 1 = \lambda(c_0) - 1 \neq 0$$

where $\lambda(c_0)$ is the eigenvalue of the orbit of $z(c_0)$, we can apply the Implicit Function Theorem. The latter yields open neighborhoods U of c_0 and V of $z(c_0)$ and a map $z\colon U \to V$ such that $Q(z(c), c) = 0$ on U. As Q is holomorphic, so is z.[4] The last equation reads $p_c^m(z(c)) = z(c)$, i.e. $z(c)$ is a periodic point for p_c of period m. For c close to the original point, $z(c)$ is attractive for p_c. This proves that c_0 is an interior point of $\mathcal{M}$.
Let H denote the connected component around c_0. By proposition 3.2, the family $f_k(c) = p_c^{km}(0)$ is bounded, hence normal on H. Let f denote a limit function of $(f_k)_{k\geq 1}$. By attractiveness, f takes value $z(c)$ in a neighborhood of c_0, and therefore solves $Q(f(c), c) = 0$ in that neighborhood. By the identity principle, this equation holds on all of H, thus providing periodic points $f(c)$ for every parameter $c \in H$ of period some multiple of m. However, since H is connected, each of these periodic points actually has period m. If one of the $f(c)$ was a repelling periodic point, then the sequence $f_k(c)$ could not have converged to that point unless it was eventually constant to $f(c)$. Such a c then is a root of $p_c^m(p_c^{km}(0)) - p_c^{km}(0)$ (viewed as a polynomial with variable c) for some k. Hence, there are at most countably many, but since the eigenvalue $\lambda(c)$ of the periodic orbit depends analytically on c, there are, in fact, none. This gives a universal bound for the eigenvalue $|\lambda(f(c))| \leq 1$ on H. But as a holomorphic map, λ is open. We conclude $|\lambda(f(c))| < 1$ on H, i.e. every periodic orbit is attractive as desired. □

Define

$$H_n = \{c \mid p_c \text{ has an attractive orbit of period } n\}$$

Since any bounded (super-) attractive Sullivan domain contains the tail of $\mathrm{Orb}^+(0)$ we see that p_c cannot have two of those. Consequently, $H_n \cap H_m = \emptyset$ whenever $n \neq m$. The last theorem immediately implies that each H_n is a union of connected components of the interior of the Mandelbrot set. In the case $d = 2$ the set H_1 is the main body of $\mathcal{M}$ and we know its structure very well.

Corollary 3.6. *If $d = 2$, then H_1 is a cardioid (in particular connected). Its boundary is contained in $\partial\mathcal{M}$.*

Proof. The parameter c is in H_1 if and only if p_c has an attractive fixed point. The two fixed points are

$$z_c^{\pm} = \frac{(1 \pm (1 - 4c)^{1/2})}{2}$$

and each has eigenvalue $\lambda(c) = 2z_c^{\pm}$. Hence, we can rewrite

$$H_1 = \{c \mid |\lambda(c)| < 1\} = \{\lambda/2 - \lambda^2/4 \mid |\lambda| < 1\}$$

This quadratic equation describes a cardioid, proving the first statement. Being connected, it follows from the last theorem that H_1 is exactly a connected component, thus implying the second statement. □

Recall that p_c is expanding on its Julia set if and only if $J \cap \overline{\mathrm{Orb}^+(0)} = \emptyset$ (cf. theorem 2.34). From the definition of the Sullivan domains, we see that this never is the case if a parabolic domain or a Siegel disk exist (remember that Herman rings do not exist at all). Thus, p_c is expanding on its Julia set at most and even exactly if there is an attractive or a superattractive domain. This motivates the following definition.

[4] Analyticity in the Implicit Function Theorem follows from analyticity in the Inverse Function Theorem. That the inverse function is analytic follows from the argument principle, with which we can find a power series of the inverse map. Namely, if f has non-vanishing derivative on the set $\{|w| \leq R\}$, then its inverse near z_0 is

$$g(z) = \sum_{n=0}^{\infty} \Big(\frac{1}{2\pi i}\int_{|w|=R} w \frac{f'(w)}{(f(w) - z_0)^{n+1}} dw\Big)(z - z_0)^n$$

Definition 3.7. *A connected component of one of the H_n is called a hyperbolic component of $\mathcal{M}$. The polynomial p_c is called hyperbolic if c lies in a hyperbolic component of $\mathcal{M}$.*

Remark 3.8. *Since $W^s(\infty)$ is a superattractive domain, p_c is expanding on its Julia set if $0 \in W^s(\infty)$ and thus, p_c should also be called hyperbolic if $c \notin \mathcal{M}$. However, in this framework we are mainly interested in the other case, and for simplicity we neglect calling p_c hyperbolic if $c \notin \mathcal{M}$.*

The main result concerning hyperbolic components is that for degree two each one is conformally equivalent to the unit disk. Thus, the interior of $\mathcal{M}$ actually behaves rather nicely. We will see later that it is the boundary that is very hard to get a hold on. The next result is due to Douady in degree two. A proof for this case can be found in [21, p. 29] or [5, p. 134]. The statement for higher degrees is in [9, p. 26].

Theorem 3.9. *If $d = 2$, any hyperbolic component H is conformally equivalent to the unit disk. The equivalence is given by*

$$\phi\colon H \to D, c \to \lambda_c$$

where λ_c is the eigenvalue of the attractive orbit of p_c. Moreover, the map extends continuously to ∂H. If $d \geq 3$, the map ϕ is not a conformal equivalence but a covering map of degree $d-1$ ramified over 0 and it also extends continuously to the boundary.

Hyperbolicity is closely linked to local connectedness[5] of the (filled-in) Julia set of p_c. But it can also be used to study local connectedness of the Mandelbrot set. The latter is what we are interested in. For the former consult, for example, [7]. Local connectedness of $\mathcal{M}$ (or, more precisely, $\partial\mathcal{M}$) is one of the primary questions of interest in complex dynamics. Up to date it remains an open problem. The following is widely accepted and the object of many studies.

Conjecture 3.10 (MLC)**.** *The Mandelbrot set is locally connected.*

As a topological property local connectedness can be quite difficult to prove. However, the following strong theorem of Carathéodory will allow us to reformulate the problem. As a reference, the reader may consult [19, p. 169] or [7, p. 23].

Theorem 3.11 (Carathéodory)**.** *Suppose $G \subset \overline{\mathbb{C}}$ is a simply connected domain, which is conformally equivalent to the unit disk D. Then the boundary of G is locally connected if and only if the conformal equivalence extends to a continuous map $\overline{G} \to \overline{D}$.*

Suppose now $d = 2$. With theorem 3.9 we see that each hyperbolic component has a locally connected boundary. Thus, the next conjecture is stronger than (MLC).

Conjecture 3.12 (HIM)**.** *The hyperbolic components constitute all of the interior of $\mathcal{M}$.*

In fact, (MLC) and (HIM) are equivalent. To show (MLC) $\Rightarrow$ (HIM) one needs tools related to the conformal equivalence $\Phi\colon \overline{\mathbb{C}} \setminus \mathcal{M} \to \mathbb{C} \setminus \overline{D}$, which we discussed earlier. For further reading see [7] and [21]. Furthermore, note that this equivalence increases the interest in the conjecture (MLC) even more. Because if (HIM) is true, then this says that the hyperbolic systems are dense in *all* the polynomial dynamical systems of degree two. In fact, one can then use similar techniques to get better hyperbolicity results in higher degrees, as well.

Up to non-hyperbolic components that may or may not exist we have completely classified the components of the interior. Let us now investigate the boundary of $\mathcal{M}$. For this we turn back to the general case $d \geq 2$. We first show that there are no parts of $\mathcal{M}$ that consist of a nowhere dense branch of points. More precisely, we show that the interior of the Mandelbrot set is dense.

[5] $G \subset \overline{\mathbb{C}}$ is locally connected at $z \in G$ if for any open set $z \in V \subset G$ there exists a connected open set U with $z \in U \subset V$. G is locally connected if it is locally connected at every point.

Proposition 3.13. *$\partial\mathcal{M}$ is contained in the closure of $\{c \mid p_c$ has a superattractive periodic orbit$\}$. Consequently, $\mathcal{M}$ has dense interior.*

Proof. Let U be an open set with $U \cap \partial\mathcal{M} \neq \emptyset$ and $0 \notin U$. Take a branch of $z^{1/d}$ defined on all of U. Suppose for contradiction U does not contain any parameter c for which 0 is periodic under p_c. Then $p_c^n(0) \neq (-c)^{1/d}$ for all $n \geq 0$ and all $c \in U$ since otherwise $p_c^{n+1}(0) = 0$. Set

$$f_n(c) = \frac{p_c^n(0)}{(-c)^{1/d}}$$

and observe that the family $(f_n)_{n\geq 0}$ omits $0, 1, \infty$ and is therefore normal on U. But $(f_n)_{n\geq 0}$ cannot be equicontinuous since U contains points for which the sequence $(p_c^n(0))_{n\geq 0}$ is bounded as well as points for which it is unbounded, a contradiction. That the interior is dense follows from theorem 3.5. $\square$

In the remaining of the chapter we provide a short exhibition of further properties of the boundary. The results are taken from [7], [21] and [23].
As a consequence of theorem 3.9 each hyperbolic component H contains a unique point $\phi^{-1}(0)$ that corresponds to a superattractive periodic orbit. We call it the *center* of H. More precisely, the parameters c for which p_c has a superattractive periodic orbit are in a 1-1 correspondence with the center of hyperbolic components. The last proposition tells us that near any boundary point there must be smaller and smaller hyperbolic components scattered around that point.
Similarly, $\phi^{-1}(1)$ consists of d points on the boundary of H. It can be shown that these d points correspond to systems p_c that have a parabolic periodic point. In addition, any parameter c for which p_c has a parabolic periodic point is such a boundary point of some hyperbolic component. Moreover, unless the period of the hyperbolic component is one (in the sense of the definition of the sets H_n), then the parabolic orbit from exactly one of the parameters in $\phi^{-1}(1)$ has eigenvalue 1 and we call this parameter the *root* of H. Whenever boundaries of two hyperbolic components intersect, the intersection is a single point and it is the root of one of the components. Likewise, a root always lies at an intersection point. Most importantly, the following statement holds.

Proposition 3.14. *The roots of hyperbolic components are dense in $\partial\mathcal{M}$.*

That at every root there is a new hyperbolic component attached confirms the above observation that smaller and smaller hyperbolic components scatter the boundary.
Another important class of boundary points are the parameters c for which 0 is strictly preperiodic for p_c. These are called *Misiurewicz points*. Such a point is a root of $p_c^{n+k}(0) - p_c^k(0)$ (viewed as a polynomial with variable c) for some $n, k \in \mathbb{N}$. Hence, there are only countably many. As for roots, the following holds.

Proposition 3.15. *The Misiurewicz points are dense in $\partial\mathcal{M}$.*

The study of the various types of parameters in $\partial\mathcal{M}$ is closely linked to the study of the conjectures (MLC) and (HIM). But we can also ask ourselves what the boundary looks like as a geometric object. The last two chapters are dedicated to this question. We will see in two different ways that the Mandelbrot set is self-similar, thus confirming our intuitive perception.

4 An Application of Polynomial-like Maps

In this chapter we introduce the notion of polynomial-like maps, which are a powerful tool for studying dynamics of polynomials. The advantage will be that we can make small perturbations of a polynomial in the space of holomorphic functions instead of just the space of polynomials. This allows for more flexibility if we want to enforce a certain condition on the perturbation. We will first give an example of the strength of polynomials and give a sharp bound on the number of non-repelling periodic orbits a polynomial can have. Then we use the newly developed tools to study the self-similarity of the Mandelbrot set. This chapter is based on the work by Douady and Hubbard in [8].

Definition 4.1. *A polynomial-like map of degree d is a triple (U, U', f) where U and U' are open subsets of $\mathbb{C}$ biholomorphic to disks with U' relatively compact in U, and $f\colon U' \to U$ is a proper analytic map of degree d.*

Remark 4.2. *Note that we can use relative compactness of U' to copy the proof of lemma 2.18, which said that $R\colon \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ is a branched covering map, almost word for word. Hence, a polynomial-like map is a branched covering map, as well, and in the above definition the degree is a well-defined notion.*

The notion of polynomial-like maps is (as the name suggests) closely linked to polynomials, as seen further below. We can introduce similar objects of interest.

Definition 4.3. *The set $K_f = \bigcap_{n\geq 0} f^{-n}(U')$ is called filled-in Julia set and its boundary the Julia set.*

Lemma 4.4. *The filled-in Julia set is compact.*

Proof. Suppose there was a sequence $(z_n)_{n\geq 0} \subset K_f$ with a limit point z outside of K_f. Then there is some iterate $f^N(z) \notin U'$ with $N \geq 1$. Since $f^N(z_n)$ converges to $f^N(z)$ but also stays in U', it must converge to the boundary of U'. Now $(f^N(z_n))_{n\geq 0}$ is a sequence in U' with no convergent subsequence in U'. As f is proper, it maps such a sequence to a sequence with no convergent subsequence in U. Thus, $f^{N+1}(z_n)$ converges to the boundary of U. However, it is also a sequence in U' as $z_n \in K_f$, and we see that U' cannot be relatively compact in U, a contradiction. □

If f is the restriction of a polynomial p to U', then this definition basically agrees with the one that has already been introduced. Indeed, $z \in K_f$ implies that $\mathrm{Orb}^+(z)$ is bounded since it never leaves U', i.e. $z \notin W^s(\infty)$. Thus, K_f is a subset of the filled-in Julia set $K(p)$. However, U' could be too small to contain the entire filled-in Julia set $K(p)$, which is why, in general, the reverse inclusion $K(p) \subset K_f$ does not hold. In fact, K_f consists exactly of those connected components of $K(p)$ whose iterates are always a subset of U'. In this example, the Julia set also is a subset of the usual Julia set.
Now suppose f is any polynomial-like map, not necessarily the restriction of a polynomial. Note that K_f could be empty! An example of this is a polynomial restricted to a proper subset of one of the components of its Fatou set. To circumvent this problem, we think of U' to be usually "quite large". With K_f being non-empty, the filled-in Julia set satisfies the same relation to critical points as it did for polynomials.

Proposition 4.5. *K_f is connected if and only if all the critical points belong to K_f.*

We base our proof on [20].

Proof. We claim that the connected components of $f^{-n}(U')$ are all biholomorphic to disks. Suppose for now the claim is true. Since $f^n\colon f^{-n}(U') \to U'$ is analytic and proper, it is a branched covering map[6].

[6]This follows just as in lemma 2.18. Note that relative compactness of U' is enough to substitute compactness of $\overline{\mathbb{C}}$ since we do not need an accumulation point to be in the domain of definition to apply the identity principle.

Applying the Riemann-Hurwitz formula to this covering map yields (in the next few lines χ is the Euler characteristic, not the straightening map)

$$\sum_{p \in f^{-n}(U')} (e_p - 1) = d \cdot \chi(U') - \chi(f^{-n}(U')) = d - \chi(f^{-n}(U'))$$

Note that applying the Riemann-Hurwitz formula to the covering map $f\colon U' \to U$ also yields

$$\sum_{p \in U'} (e_p - 1) = d \cdot \chi(U) - \chi(U') = d - 1$$

Thus, we get

$$\chi(f^{-n}(U')) = 1 \iff \sum_{p \in f^{-n}(U')} (e_p - 1) = \sum_{p \in U'} (e_p - 1)$$

i.e. $f^{-n}(U')$ consists of a single component biholomorphic to a disk if and only if all the critical points belong to $f^{-n}(U')$. In particular, if all critical points belong to $K_f = \bigcap_{n \geq 0} f^{-n}(U')$, then K_f is the non-empty intersection of disks and is itself connected. Conversely, if not all the critical points belong to K_f, then $f^{-n}(U')$ has two or more components for each $n \geq N$ and some N. Since K_f contains a point in each of these components, it must be disconnected.
Let us now prove the claim. Suppose first that $\partial U'$ does not meet any critical orbit. In particular, $\partial U'$ contains no critical values of any f^n, $n \geq 1$, and since the absence of critical values corresponds to local injectivity, the boundary of the set $f^{-n}(U')$ consists of a collection of closed Jordan curves. Suppose there was a component V that it not simply connected. Then the set $\mathbb{C} \setminus V$ has a bounded component, denoted W. Again, since f^n is analytic and proper, it maps ∂W into $\partial U'$ and would map W onto $\overline{\mathbb{C}} \setminus \overline{U'}$ if it was well-defined on W. Since this cannot happen, there is a point $z \in W$ such that the orbit of z leaves U' before time n, i.e. $f^k(z) \in U \setminus \overline{U'}$ for some $k < n$. But then we find a small disk W_2 around z with $W \subset \mathbb{C} \setminus f^{-k}(U')$ and we can repeat the process. As the power of the iterate decays each time we make this argument, we reach $k = 0$ eventually, yielding a contradiction.
Suppose now $\partial U'$ meets a critical orbit. Since K_f is compact, it is a positive distance away from $\partial U'$. Pick $K_f \subset U_2' \subset U'$ such that $\partial U_2'$ avoids any critical orbit. We then replace the polynomial-like map (f, U, U') with (f, U, U_2'). The filled-in Julia set of the new map clearly is the same as before. ☐

Similarly, other results from the first chapter can be adapted just alike.

Corollary 4.6. *For any attractive periodic point p, there is a critical point in its immediate attractive basin.*

Proof. In the proof of proposition 2.22 we only made use of properties of branched coverings and did not need that R was defined on all of $\overline{\mathbb{C}}$. ☐

Corollary 4.7. *A polynomial-like map of degree d has at most $d-1$ attractive periodic orbits.*

To state the first important theorem in the theory of polynomial-like maps we need the following notion of equivalence.

Definition 4.8. *Two polynomial-like maps f and g are quasi-conformally equivalent if there exists a homeomorphism h from a neighborhood of K_f to a neighborhood of K_g such that $h \circ f = g \circ h$, and h is quasi-conformal. If, in addition, $\overline{\partial} h = 0$ on K_f, then f and g are said to be hybrid equivalent.*

Remark 4.9. *The del-bar operator $\overline{\partial} = \frac{\partial}{\partial \overline{z}}\, d\overline{z}$ needs to be understood in a distributional sense. Namely, $\overline{\partial} h = 0$ on K_f means that*

$$\int_V h \cdot (\frac{\partial}{\partial \overline{z}} \phi)\; d\overline{z} = 0$$

for any test function ϕ with compact support in K_f, where V denotes the neighborhood of K_f from the definition above. The existence of $\overline{\partial}$ as a distribution follows from h being quasi-conformal. By Weyl's lemma, h is holomorphic on K_f up to a set of measure zero. The Julia set ∂K_f could have positive measure, but examples for that are not known. Thus, in every known case, $\overline{\partial} h = 0$ reduces to saying that h is holomorphic on the interior of K_f.

The next theorem justifies the name polynomial-like and also why these objects are useful for us. It is an application of the measurable Riemann mapping theorem. The full proof can be found in [8, p. 296-303] and proofs of the first statement in [5, p. 99-100] or [3, p. 134-135].

Theorem 4.10. *Every polynomial-like map of degree d is hybrid equivalent to a polynomial of degree d. Moreover, if K_f is connected, then the polynomial is unique up to conjugation with an affine map.*

Thus, a polynomial-like map has the same dynamics as a polynomial near its filled-in Julia set. An application is a sharpening of the count of non-repelling periodic orbits for a polynomial.

Proposition 4.11. *If p is a polynomial of degree d, then it has at most $d - 1$ non-repelling periodic orbits.*

We give the proof found in [5, p. 100] with a minor modification.

Proof. Let N denote the set of non-repelling periodic points of p. Since N is finite as of proposition 2.24, we can construct a polynomial q that vanishes on N. Furthermore, by taking a polynomial of higher degree we can pick q such that

$$\sum_{j=1}^{m} Re(\frac{q'(z_j)}{p'(z_j)}) < 0$$

for every neutral periodic orbit $\{z_1, ..., z_m\}$. Define $f = p + \epsilon q$ for $\epsilon > 0$. Since q vanishes on N, every non-repelling periodic orbit of p is a periodic orbit for f. If ϵ is small enough, any attractive orbit for p is also attractive for f. Moreover, by shrinking ϵ even further, any neutral orbit of p becomes an attractive orbit for f, which we see as follows: Let $\{z_1, ..., z_m\}$ denote a neutral orbit of p, λ its eigenvalue with respect to p and μ its eigenvalue for f. Using

$$\sum_{j=1}^{m} \log|p'(z_j)| = \log \prod_{j=1}^{m} |p'(z_j)| = \log|\lambda| = 0$$

and

$$\log|1 + \epsilon \frac{q'(z_j)}{p'(z_j)}|^2 = \log(1 + 2\epsilon Re(\frac{q'(z_j)}{p'(z_j)}) + O(\epsilon^2)) = \epsilon Re(\frac{q'(z_j)}{p'(z_j)}) + O(\epsilon^2)$$

we get

$$2 \cdot \sum_{j=1}^{m} \log|f'(z_j)| = 2 \cdot \sum_{j=1}^{m} \log|p'(z_j)| + \sum_{j=1}^{m} \log|1 + \epsilon \cdot \frac{q'(z_j)}{p'(z_j)}|^2 = \epsilon \cdot \sum_{j=1}^{m} Re(\frac{q'(z_j)}{p'(z_j)}) + O(\epsilon^2) < 0$$

meaning

$$|\mu|^2 = \exp(2 \cdot \sum_{j=1}^{m} \log |f'(z_j)|) < 1$$

Take $U \subset \mathbb{C}$ large (and connected and simply connected) such that $U' = f^{-1}(U)$ encompasses all finite critical points of p. Since ∞ is an attractive fixed point of f, when thought of as a map on the Riemann sphere, we can pick U so large that $U' \subset U$. We claim that f is a polynomial-like map of the same degree d. Indeed, if $z \in U$ has inverse images $z_1, \ldots, z_n$ under p, then it will have inverse images $w_1, \ldots, w_n$ close to the z_i under f because ϵ was chosen to be very small. On the other hand, it cannot have more inverse images under f by the same argument. We conclude using corollary 4.7. □

This proof is a good example of the strength of polynomial-like maps. Before, we could only turn half of the neutral orbits into attractive ones because the perturbed map needed to be a polynomial (see proposition 2.24).

This short exhibition concludes our motivation for polynomial-like maps. We proceed to the main interest of this chapter. From now on we restrict our attention to the case of degree two, i.e. every polynomial-like map in consideration will be of degree two. The goal is to give one (of several possible) explanations of the self-similarity of $\mathcal{M}$. Namely, that if one zooms into a computer generated picture of $\mathcal{M}$, it will look like before zooming in. More specifically, we will show that these small subsets, which we see by zooming in, can be mapped quasi-conformally onto $\mathcal{M}$. Since a quasi-conformal map does not distort its image very much in an intuitive sense, this explains the small copies of $\mathcal{M}$.
Our approach is to first mimic $\mathcal{M}$ with a similar set that arises from a family of certain holomorphic maps (just like $\mathcal{M}$). Then, under certain hypothesis, we will naturally get a quasi-conformal map between two mimicries. Lastly, in a special case, one of the two mimicries will be such a subset of $\mathcal{M}$ and the second one will be $\mathcal{M}$ itself. To construct the mimicries we need the notion of an analytic family.

Definition 4.12. *Let Λ be a complex manifold and $F = (f_\lambda \colon U'_\lambda \to U_\lambda)_{\lambda \in \Lambda}$ be a family of polynomial-like maps. Set $U = \{(\lambda, z) \mid z \in U_\lambda\}$, $U' = \{(\lambda, z) \mid z \in U'_\lambda\}$ and $f(\lambda, z) = (\lambda, f_\lambda(z))$. We say F is an analytic family if*

1. *U and U' are homeomorphic over Λ to $\Lambda \times D$,*
2. *The projection from the closure of U' in U to Λ is proper and*
3. *$f \colon U' \to U$ is analytic and proper.*

In our framework we will always deal with families parametrized by an open subset $\Lambda \subset \mathbb{C}$ homeomorphic to a disk. Given an analytic family of polynomial-like maps of degree two, we naturally get a map $\chi \colon \Lambda \to \mathbb{C}$ specified by the property that f_λ is hybrid equivalent to $z^2 + \chi(\lambda)$. χ is called the *straightening map*. A priori, it depends on a choice. Fortunately, theorem 4.10 enables us to circumvent this problem.

Corollary 4.13. *Suppose $c_1,\ c_2 \in \mathcal{M}$. If p_{c_1} and p_{c_2} are hybrid equivalent, then $c_1 = c_2$.*

Proof. By hypothesis p_{c_1} is hybrid equivalent to both p_{c_2} and itself. By the second part of the theorem, p_{c_1} and p_{c_2} are conjugate by an affine map. The only affine map conjugating two polynomials of this form is the identity. □

Define $M_F = \{\lambda \mid K_{f_\lambda} \text{ is connected}\}$. This set is the mimicry of $\mathcal{M}$ mentioned earlier. A posteriori, this corollary shows that the straightening map is independent of a choice at least inside M_F. We can say something about the regularity of χ. Namely, it is continuous on all of Λ (see [8, p. 308]). The proof proceeds as follows.
We first need to introduce a new notion for periodic points. Given an analytic family and a periodic point p of f_{λ_0} with eigenvalue in the unit circle minus the point 1, each map close to f_{λ_0} has a periodic point

of the same period as p. We say p is *persistently non-hyperbolic* if each of the periodic points of the close maps has eigenvalue of absolute value one, i.e. we cannot make it hyperbolic by small perturbations.[7] Note that the eigenvalue of the close periodic points must actually be exactly the same as the one of p by analytic dependence of the eigenvalue. The set $H(F)$ denotes the set of parameters λ that have an open neighborhood in Λ such that for every λ' in that neighborhood, $f_{\lambda'}$ has a hyperbolic or persistently non-hyperbolic periodic point.[8] Mañé, Sad and Sullivan proved that $H(F)$ is open and dense in Λ and that for two sufficiently close parameters $\lambda_1, \lambda_2 \in H(F)$, the Julia sets and hence, filled-in Julia sets of f_{λ_1} and f_{λ_2} are homeomorphic (see [17, p. 199]).[9] It follows that a connected component of $H(F)$ either belongs to the interior of M_F or the complement of M_F. We can say even more.

Proposition 4.14. *$H(F) = \Lambda \setminus \partial M_F$.*

We closely follow [8, p. 309].

Proof. We already know one inclusion and only need to prove $\Lambda \setminus \partial M_F \subset H(F)$. We will first show that the interior of M_F is a subset of $H(F)$ and then that $\Lambda \setminus M_F$ is, too.
Suppose for contradiction there is some λ_0 in the interior of M_F but not in $H(F)$. Unraveling the definition of $H(F)$, we find a neighborhood W of λ_0 such that each f_λ, $\lambda \in W$, has a periodic point $z(\lambda)$ of period k and eigenvalue $\rho(\lambda)$, where z and ρ depend analytically on λ and ρ is non-constant. Furthermore, let $\omega(\lambda)$ denote the critical point of f_λ. Take a sequence $\lambda_n \to \lambda_0$ in W such that $|\rho(\lambda_n)| < 1$ for all n. By corollary 4.6, each $\omega(\lambda_n)$ belongs to the basin of $z(\lambda_n)$. Hence, for each n, there is some $0 \le i(n) \le k-1$ such that

$$f_{\lambda_n}^{km+i(n)}(\omega(\lambda_n)) \to z(\lambda_n) \text{ as } m \to \infty$$

We may pick a subsequence of λ_n such that $i(n)$ is constant i. Now define

$$g_m(\lambda) = f_\lambda^{km+i}(\omega(\lambda))$$

Since λ_0 was in the interior of M_F, we can assume $W \subset M_F$ by shrinking if necessary. By definition of M_F, $\omega(\lambda)$ belongs to the filled-in Julia set of f_λ for $\lambda \in W$. In particular, $g_m(\lambda) \in U'_\lambda$. Given a compact set $K \subset W$, the set $\{(\lambda, z) \mid z \in \overline{U'_\lambda},\ \lambda \in K\}$ is compact by the second property in the definition of an analytic family. Therefore, $g_m(\lambda)$ is bounded on compact sets in W, hence normal on W. If g denotes any limit function, then $g(\lambda_n) = z(\lambda_n)$ for every n, and therefore $g = z$ by the identity principle. Thus, $g_m(\lambda) \to z(\lambda)$ for any λ and not just on a subsequence. However, as a holomorphic function, ρ also takes values of absolute value strictly greater than one. For those, $z(\lambda)$ is repelling and $g_m(\lambda)$ could not converge to it, a contradiction. This concludes the first inclusion.
Now suppose $\lambda \in \Lambda \setminus M_F$. Then the critical point ω of f_λ lies in $U'_\lambda \setminus K_{f_\lambda}$. As in the case of polynomials, this implies that f_λ is expanding on its Julia set ([8, p. 296]). In particular, there are no periodic points with eigenvalue of absolute value one. This shows the second inclusion and finishes the proof. □

Douady and Hubbard used that the area of K_{f_λ} is continuous in $H(F)$ to show that the straightening map is continuous in $H(F)$ ([8, p. 310]). The last step is to prove continuity on ∂M_F by approximation using density of $H(F)$ ([8, p. 313]). That concludes continuity of χ.
We will mostly be interested in how χ behaves on M_F. On the interior of this set the straightening map even is analytic since it is on $H(F)$ ([8, p. 313]). Moreover, by restricting to M_F we get better topological properties, as well.

Lemma 4.15. *If χ is not constant and M_F is compact, then the restriction $\chi\colon M_F \to \mathcal{M}$ is surjective.*

[7] The exact definition differs because it needs to include the case where p has eigenvalue 1. For details see [17, p. 198].

[8] In the case of polynomials of the form $z \to z^d$, the set $H(f)$ is exactly the interior of the Mandelbrot set, i.e. the parameters with persistently non-hyperbolic periodic points correspond to the non-hyperbolic components.

[9] Even though [17] does not deal with polynomial-like maps, the proofs proceed analogously.

Proof. It is shown in [8, p. 326] that χ is topologically holomorphic over $\mathcal{M}$, which, roughly speaking, means that it is locally a covering map. More precisely, for every $p \in M_F$ there exists an open neighborhood V of p and an open neighborhood U of $\chi(p)$ such that the restriction $\chi\colon V \to U$ is proper and surjective ([8, p. 322]). By compactness, $(V_p)_{p\in M_F}$ has a finite subcover of M_F, which we denote by $V_1, \ldots, V_n$. Note that the boundary of $\bigcup_{1\le j\le n} V_j$ lies in $\Lambda \setminus M_F$. Moreover, as a proper map, χ maps each ∂V to ∂U. Using $\chi^{-1}(M) = M_F$, we see that the boundary of $W = \bigcup_{1\le j\le n} U_j$ is disjoint to $\mathcal{M}$. The set W must contain all of $\mathcal{M}$ since otherwise we could write $\mathcal{M}$ as the disjoint union of open sets $\mathcal{M} \cap W$ and $\mathcal{M} \cap \mathbb{C} \setminus \overline{W}$, contradicting connectedness of $\mathcal{M}$. Since each $\chi\colon V \to U$ was surjective, we conclude that χ is surjective onto $\mathcal{M}$. Lastly, use the equality $\chi^{-1}(M) = M_F$ once more. □

Remark 4.16. *The condition in the last lemma can be visualized using the trivial example. If $f_\lambda(z) = z^2 + \lambda$ and Λ is some subset of $\mathcal{M}$, then χ is the identity map restricted to Λ, hence not surjective. In order to satisfy the hypothesis of the lemma, Λ needs to contain all of $\mathcal{M}$ and the statement of the lemma becomes, in fact, trivial.*

From now on we will always assume that χ is not constant. Clearly, this is not very restrictive. Note that as χ is holomorphic on the interior of M_f, we always have $\chi^{-1}(\partial\mathcal{M}) \subset \partial M_f$. Indeed, if there was a point $\lambda \in M_f \setminus \partial M_f$ with $\chi(\lambda) \in \partial\mathcal{M}$, then for a small open neighborhood $U \subset M_f \setminus \partial M_f$ of λ the set $\chi(U)$ also is open, contradicting $\chi(M_f) \subset \mathcal{M}$. Therefore, if M_f is compact and the previous lemma holds, then $\chi\colon M_F \to \mathcal{M}$ is a branched covering map.

Definition 4.17. *We call the family F Mandelbrot-like if M_f is compact and the degree of $\chi\colon M_F \to \mathcal{M}$ is one, i.e. χ restricts to a homeomorphism $M_F \to \mathcal{M}$.*

The straightening map will become the quasi-conformal map from the mimicry of $\mathcal{M}$ into $\mathcal{M}$. To show quasi-conformality we will use the λ-lemma. In this form it was first stated and proved in [17, p. 193, 201]. It will also play an important role in the next chapter.

Theorem 4.18 (λ-lemma)**.** *Let $i_\lambda\colon X \to \overline{\mathbb{C}}, \lambda \in D$, where $X \subset \overline{\mathbb{C}}$ is any subset and D is a disk centered at the origin. Suppose each i_λ is injective, $i_0 = id_X$ and that for fixed $z \in X$ the map $i_\lambda(z)\colon D \to \overline{\mathbb{C}}$ depends analytically on λ. Then there is an extension to a family of maps $j_\lambda\colon \overline{X} \to \overline{\mathbb{C}}$, $\lambda \in D$, which is jointly continuous and such that each j_λ is quasi-conformal.*

In order to apply it, we extend the notion of an analytic family of maps and pair it with the well known concept of continuous paths. Let $I = [0, 1]$ be the unit interval and $F = (f_{s,\lambda}\colon U'_{s,\lambda} \to U_{s,\lambda})_{s\in I,\lambda\in\Lambda}$ be a family of polynomial-like maps of degree two. We replace U in definition 4.12 by $U = \{(s, \lambda, z) \mid z \in U_{s,\lambda}\}$, similarly for U', and f by $f(s, \lambda, z) = (s, \lambda, f_{s,\lambda}(z))$. We again require U and U' to be homeomorphic over $I \times \Lambda$ to $I \times \Lambda \times D$ and the projection of the closure of U' in U to Λ to be proper. Similarly, f is supposed to be proper, analytic in (λ, z) and continuous in s. Furthermore, assume that for each $s \in I$ the analytic family $F_s = (f_{s,\lambda})_{\lambda\in\Lambda}$ is Mandelbrot-like and that there exists a compact set A in Λ such that for every s we have $M_{F_s} \subset A$.

Definition 4.19. *If these conditions are satisfied, then F is called a continuous path of Mandelbrot-like families connecting F_0 and F_1.*

We are ready to invoke quasi-conformality.

Proposition 4.20. *Suppose $F = (f_\lambda\colon U'_\lambda \to U_\lambda)_{\lambda\in\Lambda}$ and $G = (g_\lambda\colon U'_\lambda \to U_\lambda)_{\lambda\in\Lambda}$ are two Mandelbrot-like families parametrized by the same Λ. If F and G can be connected by a continuous path of Mandelbrot-like families, then the homeomorphism $\chi = \chi_G^{-1} \circ \chi_F\colon M_F \to M_G$ is quasi-conformal.*

We base our proof on [8, p. 329].

Proof. We divide the path connecting F and G into finitely many small pieces. Since the composition of two quasi-conformal maps is again quasi-conformal, it suffices to show the proposition for families F and G on the same segment. Let $(H_s)_{s\in I}$ denote the path connecting F and G. By definition, there exists a compact set A in Λ such that for every s we have $M_{H_s} \subset A$. Consider the family

$$F_t = (f_\lambda + t \cdot (g_\lambda - f_\lambda))_{\lambda\in\Lambda'}$$

for $t \in D_R(0) \subset \mathbb{C}$ and $A \subset \Lambda' \subset \Lambda$. If the segments chosen in the beginning are small enough and $R > 1$ is small, then $M_{F_t} \subset A$ still is compact. Thus, each straightening map $\chi_{F_t} \colon M_{F_t} \to \mathcal{M}$, associated to the family F_t, is a branched covering map. Since the degree is locally constant, each χ_{F_t} is, in fact, a homeomorphism. Thus, (for appropriate domains) the families F_t are Mandelbrot-like, as well. Define

$$\chi_t = \chi_{F_t}^{-1} \circ \chi_F \colon M_F \to \Lambda'$$

Since χ_t depends analytically on t, it satisfies the hypothesis of the λ-lemma (after resizing $D_R(0)$ to the unit disk). The λ-lemma tells us that each χ_t is quasi-conformal, in particular $\chi_1 = \chi_G^{-1} \circ \chi_F$ is. $\square$

We finished constructing mimicries of $\mathcal{M}$ and quasi-conformal maps between them. Now we need to investigate which mimicries are subsets of $\mathcal{M}$. Such arise if the analytic family generating M_F is similar to the family $(p_c)_{c\in\mathbb{C}}$ from which $\mathcal{M}$ itself stems.

Definition 4.21. *Let $c \in \mathcal{M}$ and suppose 0 is a periodic point of period k for p_c. We say c is tunable if there exists a neighborhood Λ of c and a Mandelbrot-like family $F = (f_\lambda \colon U'_\lambda \to U_\lambda)_{\lambda\in\Lambda}$ such that for every $\lambda \in \Lambda$ we have $0 \in U'_\lambda$ and the map f_λ is the restriction of p_λ^k to U'_λ.*

The set M_F corresponding to this analytic family is what we are looking for. Let us check that it is a subset of $\mathcal{M}$.

Lemma 4.22. *For F as in the definition of a tunable point, $M_F \subset \mathcal{M}$ and $\partial M_F \subset \partial\mathcal{M}$.*

Proof. By definition

$$K_{f_\lambda} \subset K_{p_\lambda^k} \subset K(p_\lambda^k) \subset K(p_\lambda)$$

where $K_{(\cdot)}$ is the filled-in Julia set of a polynomial-like map and $K(\cdot)$ is the usual filled-in Julia set of the map viewed as polynomial on all of $\overline{\mathbb{C}}$. Hence, if K_{f_λ} is connected, then $K(p_\lambda)$ cannot be totally disconnected and consequently must be connected. Thus, $\lambda \in \mathcal{M}$. Conversely, if K_{f_λ} is not connected, then $K(p_\lambda)$ cannot be connected and $\lambda \notin \mathcal{M}$. $\square$

We gathered all the necessary ingredients to prove self-similarity.

Corollary 4.23. *Given a tunable point c, there is a small quasi-conformal copy of $\mathcal{M}$ at c.*

Proof. Define

$$f_{s,\lambda} = s \cdot p_\lambda(z) + (1-s) \cdot p_\lambda^k(z)$$

If F denotes the analytic family from the definition of a tunable point, then $F = (f_{0,\lambda})_{\lambda\in\Lambda}$. Moreover, we define $G = (p_\lambda)_{\lambda\in\Lambda} = (f_{1,\lambda})_{\lambda\in\Lambda}$. We see that $(f_{s,\lambda} \colon U'_{s,\lambda} \to U_{s,\lambda})_{s\in I,\lambda\in\Lambda}$ is a continuous path of Mandelbrot-like families for appropriate $U'_{s,\lambda}$ and $U_{s,\lambda}$. Thus, $\chi = \chi_G^{-1} \circ \chi_F \colon M_F \to M_G$ is quasi-conformal by proposition 4.20. By corollary 4.13, χ_G is the identity and we conclude that χ_F is quasi-conformal. The statement follows from lemma 4.22. $\square$

All that is left to do is assert the existence of tunable points. Moreover, since we do not want a single quasi-conformal copy of $\mathcal{M}$, there hopefully are infinitely many tunable points clustering $\partial\mathcal{M}$. Douady and Hubbard provided the necessary proof, see [8, p. 332-337].

Theorem 4.24. *Given any Misiurewicz point c, there is sequence of tunable points converging to c.*

Since Misiurewicz points are dense in $\partial\mathcal{M}$ (see proposition 3.15), we have reached the goal of this chapter. We showed that regardless of where we close in on the Mandelbrot set, there will be a small quasi-conformal copy of it. In a sense, this phenomenon resembles self-similarity. However, there are various definitions of the latter. Often, self-similarity and exhibiting fractal nature are not distinguished. We will conclude this thesis by proving that $\partial\mathcal{M}$ admits that kind of self-similarity, as well.

5 The Mandelbrot Set as a Fractal

5.1 The Hausdorff Dimension of $\partial\mathcal{M}$

Let us begin this chapter by introducing the Hausdorff dimension from scratch. Roughly speaking, it measures how fine a cover must be to cover the set most efficiently. Suppose X is any metric space, $A \subset X$ any subset, and s and δ are two positive real numbers. We define

$$H_s^\delta(A) = \inf\{\sum_{U \in \mathcal{U}} \operatorname{diam}(U)^s \mid \mathcal{U} \text{ is a cover of } A \text{ of sets of diameter at most } \delta\}$$

Clearly, $H_s^\delta(A)$ is increasing as δ decreases. By monotonicity, the limit

$$H_s(A) = \lim_{\delta \to 0} H_s^\delta(A) = \sup_{\delta > 0} H_s^\delta(A) \in [0, \infty]$$

exists.

Lemma 5.1. *For each s and δ both H_s^δ and H_s define a measure.*

Proof. In both cases we only need to show subadditivity. Suppose $A \subset \bigcup_{k \geq 1} A_k$ and $\epsilon > 0$. For each k pick a cover $\mathcal{U}_k$ of A_k of sets of diameter at most δ with

$$\sum_{U \in \mathcal{U}_k} \operatorname{diam}(U)^s < H_s^\delta(A_k) + 2^{-k}\epsilon$$

Since $\bigcup_{k \geq 1} \mathcal{U}_k$ is a cover of A we get

$$H_s^\delta(A) \leq \sum_{k \geq 1} H_s^\delta(A_k) + \epsilon$$

This proves that H_s^δ is a measure. From this it follows immediately that H_s is a measure, too.

$$H_s^\delta(A) \leq \sum_{k \geq 1} H_s^\delta(A_k) \leq \sum_{k \geq 1} H_s(A_k)$$

The left-hand side converges to $H_s(A)$. □

Lastly, we define the *Hausdorff dimension* of A to be

$$\dim_{\mathrm{H}}(A) = \inf\{s > 0 \mid H_s(A) = 0\}$$

That $\dim_{\mathrm{H}}$ is well-defined follows from the next lemma.

Lemma 5.2. *Suppose* $0 < t < s < \infty$. *Then*
1. $H_t(A) < \infty \Rightarrow H_s(A) = 0$ *and*
2. $H_s(A) > 0 \Rightarrow H_t(A) = \infty$.

Proof. For any cover $\mathcal{U}$ of A of sets of diameter at most δ we have

$$H_s^\delta(A) \leq \sum_{U \in \mathcal{U}} \operatorname{diam}(U)^s \leq \delta^{s-t} \cdot \sum_{U \in \mathcal{U}} \operatorname{diam}(U)^t$$

Taking the infimum over all covers we get

$$H_s^\delta(A) \leq \delta^{s-t} H_t^\delta(A) \leq \delta^{s-t} H_t(A)$$

Taking the limit $\delta \to 0$ shows part one. The second part is a consequence of the first. □

Let us proceed by giving a very general upper bound for the Hausdorff dimension.

Proposition 5.3. *It holds* $\dim_{\mathrm{H}}(\mathbb{C}^n) = 2n$*. Consequently, for any* $A \subset \mathbb{C}^n$ *it holds that* $\dim_{\mathrm{H}}(A) \leq 2n$*.*

We follow [26, p. 23].

Proof. The second statement is immediate once we have shown the first since a cover of $\mathbb{C}^n$ is also a cover of A. For the first part we need to show $H_{2n}(\mathbb{C}) > 0$ and $H_s(\mathbb{C}) = 0$ for any $s > 2n$. We first show that $0 < H_{2n}(Q) < \infty$, where $Q = [-1, 1]^n$ denotes the n complex dimensional unit cube. Let z_j, $1 \leq j \leq 2^{2n(k+1)}$ denote the points of a $2n$ dimensional mesh, which is equally distributed with distances 2^{-k} such that Q is contained in the union of cubes of edge length 2^{-k} around each z_j. Each small cube has diameter $(2n)^{1/2} \cdot 2^{-(k+1)}$. Hence, if k is so large that $(2n)^{1/2} \cdot 2^{-(k+1)} < \delta$, then

$$H^{\delta}_{2n}(Q) \leq 2^{2n(k+1)} \cdot ((2n)^{1/2} \cdot 2^{-(k+1)})^{2n} = (2n)^n$$

Taking the limit $\delta \to 0$ yields $H_{2n}(Q) \leq (2n)^n$. To show the second inequality, let λ denote the Lebesque measure and ω_{2n} the Lebesque volume of the n complex dimensional unit ball. If $\mathcal{U}$ is a cover of Q, then it also covers the unit ball and

$$\omega_{2n} \leq \sum_{U \in \mathcal{U}} \lambda(U) \leq \sum_{U \in \mathcal{U}} \operatorname{diam}(U)^{2n} \cdot \omega_{2n}$$

Thus, we always have $\sum_{U \in \mathcal{U}} \operatorname{diam}(U)^{2n} \geq 1$ and can conclude $H_{2n}(Q) \geq 1$. By the last lemma we have $H_s(Q) = 0$ for any $s > 2n$. Now cover $\mathbb{C}^n$ by countably many cubes. That $H_s(\mathbb{C}) = 0$ for any $s > 2n$ follows from subadditivity of H_s as a measure. Lastly, $H_{2n}(\mathbb{C}) \geq H_{2n}(Q) > 0$ is clear. □

We can give a lower bound as well. For this we introduce the notion of the *topological dimension*[10] of a set and show that it is a lower bound for $\dim_{\mathrm{H}}$. We define it inductively as follows. The empty set has topological dimension -1. Any other set $A \subset \mathbb{C}^n$ has topological dimension the smallest integer n such that for any point in A there are arbitrarily small neighborhoods whose boundary has topological dimension $n-1$ or less. For example, the topological dimension of $\mathbb{C}^n$ is $2n$.

Proposition 5.4. *The Hausdorff dimension is at least as large as the topological dimension.*

We base our proof on [10, p. 114].

Proof. The inductive definition of the topological dimension suggests an inductive proof. We claim that the former is at most $n-1$ whenever $H_n(A) = 0$. This suffices since $H_n(A) = 0$ holds for $n = \lceil \dim_{\mathrm{H}}(A) \rceil$, in particular.
Suppose $H_0(A) = 0$. Since H_0 is just the counting measure, A must be empty. Next, assume $H_{n+1}(A) = 0$ and $x \in A$. We claim that for almost every $r > 0$ we have $H_n(\partial D_r(x) \cap A) = 0$. Once this is shown, $\partial D_r(x) \cap A$ has topological dimension at most $n-1$ by the induction step. Then the topological dimension of A is at most n by definition. Let us now prove the claim. For any $U \subset A$

$$\overline{\int_0^\infty} \operatorname{diam}(\partial D_r(x) \cap U)^n \leq \sup_{r>0} \operatorname{diam}(\partial D_r(x) \cap U)^n \cdot \overline{\int_0^\infty} \chi_{\{\operatorname{diam}(\partial D_r(x) \cap U)^n \neq 0\}}(r) \leq \operatorname{diam}(U)^{n+1}$$

where we use the upper integral in case the integrand is not measurable. Since $H_{n+1}(A) = 0$, for every integer k there is a cover $(U_k^j)_{j \geq 1}$ of A of sets of diameter at most 2^{-k} satisfying

$$\sum_{j \geq 1} \overline{\int_0^\infty} \operatorname{diam}(\partial D_r(x) \cap U_k^j)^n \leq \sum_{j \geq 1} \operatorname{diam}(U_k^j)^{n+1} \leq 2^{-k}$$

[10] This is often called the *inductive dimension* but in our framework of working with $\mathbb{C}^n$ it agrees with the various other definitions associated with the name topological dimension.

By Fatou's lemma about interchanging limits and integrals

$$\overline{\int_0^\infty} \sum_{j\geq 1} \operatorname{diam}(\partial D_r(x) \cap U_k^j)^n \leq 2^{-k}$$

Taking the limit $k \to \infty$ and using Fatou's lemma again, we obtain

$$\overline{\int_0^\infty} \lim_{k\to\infty} \sum_{j\geq 1} \operatorname{diam}(\partial D_r(x) \cap U_k^j)^n = 0$$

Hence, for almost every $r > 0$

$$\lim_{k\to\infty} \sum_{j\geq 1} \operatorname{diam}(\partial D_r(x) \cap U_k^j)^n = 0$$

and by definition, this implies $H_n(\partial D_r(x) \cap A) = 0$ for almost every r, as desired. □

Mandelbrot used this result to give a precise definition of a fractal. Namely, a *fractal* is a set whose Hausdorff dimension is strictly greater than its topological dimension. Since $\dim_{\mathrm{H}}$ is usually thought of as a measure for the complexity of a set, this agrees with our intuitive idea of a fractal. As a general rule of thumb, the larger $\dim_{\mathrm{H}}$, the more complex it looks.
In this sense, if the boundary of $\mathcal{M}$ had a simple geometry, it would have Hausdorff dimension one (its topological dimension). However, in this chapter we will see that this is not the case. We will prove that $\partial\mathcal{M}$ has Hausdorff dimension two, that is the largest it could possibly have (being a subset of the complex plane). Furthermore, we can ask the same question for the Julia sets of p_c. The second main result of this chapter will be that the parameters for which p_c has a maximally complex Julia set (that is a Julia set of Hausdorff dimension two) are dense in $\partial\mathcal{M}$. Let us formally state these claims.

Theorem 5.5 (Fractal 1)**.** *If $U \subset \mathbb{C}$ is open and intersects $\partial\mathcal{M}$, then* $\dim_{\mathrm{H}}(U \cap \partial\mathcal{M}) = 2$.

Note that we did not just claim $\dim_{\mathrm{H}}(\partial\mathcal{M}) = 2$. What this theorem states is stronger in the sense that the boundary does not only exhibit a complex geometry in a certain region, but everywhere.

Theorem 5.6 (Fractal 2)**.** *The parameters $c \in \partial\mathcal{M}$ for which the Julia set of p_c has Hausdorff dimension two are dense in $\partial\mathcal{M}$.*

To prove these theorems we will rely on Shishikura's work [24] and expand some of the details. The first tool we need to introduce is hyperbolicity. Recall the definition of a hyperbolic set.

Definition 5.7. *Given a rational map $f \colon \overline{\mathbb{C}} \to \overline{\mathbb{C}}$, a hyperbolic set for f is a closed set $X \subset \overline{\mathbb{C}}$ with $f(X) \subset X$ and*

$$\exists C > 0,\ \mu > 1 \text{ such that } \forall n > 0 \colon |(f^n)'| \geq C\mu^n \text{ on } X$$

Remark 5.8. *The reader may note that this is not the standard definition of hyperbolicity. The latter involves a continuous df-invariant splitting of the tangent bundle of $\overline{\mathbb{C}}$ such that the exterior derivative of f is contracting on one splitting component and expanding on the other. However, in our framework each definition translates to the other one: Since $\overline{\mathbb{C}}$ is a complex manifold, the exterior derivative splits into the ∂ and the $\overline{\partial}$ operator (see [4, p. 106]), i.e. $d = \partial + \overline{\partial}$. However, since f is holomorphic $\overline{\partial} f = 0$ and so*

$$df = \partial f = \frac{\partial f}{\partial z} dz = f' \cdot (dx + idy)$$

where f' is the usual complex derivative. Now if v is a tangent vector to $\overline{\mathbb{C}}$, then we can write $v = a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y}$ for some real numbers a and b under the usual identification. Then $df(z)v = f'(z) \cdot (a + ib)$ and therefore

$$||df(z)v|| \geq C\mu||v|| \iff |f'(z)| \geq C\mu$$

This means the expansion condition on the splitting is the same as the expansion condition in the above definition.

We make the first immediate observation.

Proposition 5.9. *A hyperbolic set is always a subset of the Julia set.*

Proof. We can basically copy the proof of proposition 2.6. If a point $x \in X$ was in the Fatou set, then on a neighborhood of x the family $(f^n)_{n\geq 0}$ converges on a subsequence $(n_k)_{k\geq 0}$ uniformly on compacta to some holomorphic function f. But then we get a contradiction

$$\infty > |f'(x)| = |\lim_{k\to\infty}(f^{n_k})'(x)| \geq \lim_{k\to\infty} C \cdot \mu^{n_k} = \infty$$

□

Throughout this chapter keep the following in mind. Hyperbolicity is a rich research area itself. However, we use hyperbolic sets for only two of their properties. Firstly, they are subsets of the Julia set, which is what we are interested in. Secondly, we will need the associated persistence result, see [15, p. 410].

Proposition 5.10 (Persistence of hyperbolic sets)**.** *Suppose $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ is a holomorphic map of degree d and $X \subset \overline{\mathbb{C}}$ a hyperbolic set for f. Then there exists a neighborhood U of f in the set of holomorphic functions of the same degree such that any $g \in U$ has a hyperbolic set X_g, and there exists a homeomorphism $h_g : X \to X_g$ with $h_g \circ f = g \circ h_g$ on X and $h_f = id_X$. Moreover, $h_{(\cdot)}(z) : U \to \overline{\mathbb{C}}$ is holomorphic for any $z \in X$.*

That hyperbolic sets are subsets of the Julia set allows us to study the Hausdorff dimension of hyperbolic sets instead of the one of J. That this really is a sensible approach will become evident throughout the chapter. To simplify notation let us shortly write

$$\dim_{\mathrm{hyp}}(f) = \sup\{\dim_{\mathrm{H}}(X) \mid X \text{ is a hyperbolic set for } f\}$$

By proposition 5.9 $\dim_{\mathrm{hyp}}(f) \leq \dim_{\mathrm{H}}(J(f))$ and so $\dim_{\mathrm{hyp}}(f) = 2$ implies $\dim_{\mathrm{H}}(J(f)) = 2$.
Yet, we need a correlation between the Hausdorff dimension of some hyperbolic set and that of $\partial\mathcal{M}$.

Lemma 5.11. *Suppose we are given an open set $U \subset \mathbb{C}$ intersecting $\partial\mathcal{M}$ and a point $c_0 \in U \cap \partial\mathcal{M}$. Consider the family $f_c(z) = z^d + c$, $c \in U$. Then*

$$\dim_{\mathrm{hyp}}(f_{c_0}) \leq \dim_{\mathrm{H}}(U \cap \partial\mathcal{M})$$

This lemma will be proved towards the end of this chapter. We need one more tool to prove the theorems stated above. This lemma is a highly technical result and for the proof we refer to [24].

Lemma 5.12. *If $p_c(z) = z^d + c$, $c \in \partial\mathcal{M}$, has a parabolic periodic point with eigenvalue 1 (i.e. is the root of a hyperbolic component), then there exists a sequence $(c_n)_{n\geq 0} \subset \partial\mathcal{M}$ with $c_n \to c$ and $\dim_{\mathrm{hyp}}(p_{c_n}) \to 2$.*

This lemma tells us that for certain systems, the Julia set admits an arbitrarily complex geometry. All that is left to do is to use lemma 5.11 to translate this result to $\partial\mathcal{M}$. The proofs of the two theorems are based on the ones in [24, p. 7].

Proof of Fractal Theorem 1. Let $U \subset \mathbb{C}$ be open with $U \cap \partial\mathcal{M} \neq \emptyset$. Since roots of hyperbolic components of $\mathcal{M}$ are dense in the boundary (see proposition 3.14), we can pick a point $c \in U \cap \partial\mathcal{M}$ that satisfies the hypothesis of lemma 5.12. Using the first lemma as well, we deduce

$$\dim_{\mathrm{H}}(U \cap \partial\mathcal{M}) \geq \dim_{\mathrm{hyp}}(p_{c_n}) \to 2$$

□

The proof of the second theorem is an application of Baire's theorem.

Proof of Fractal Theorem 2. Define

$$R_n = \{c \in \partial\mathcal{M} \mid \dim_{\mathrm{hyp}}(p_c) > 2 - 1/n\}$$

Again, we use that roots of hyperbolic components are dense in $\partial\mathcal{M}$. Then, by the second lemma the sets R_n are dense, too. We claim that each R_n is open in $\partial\mathcal{M}$. If the claim holds, then

$$\bigcap_{n \geq 1} R_n = \{c \in \partial\mathcal{M} \mid \dim_{\mathrm{hyp}}(p_c) = 2\} = \{c \in \partial\mathcal{M} \mid \dim_{\mathrm{H}}(J(p_c)) = 2\}$$

is open and dense by Baire's theorem. Lastly, the claim is proved further below in corollary 5.21. □

Remark 5.13. *The set* $\{c \in \partial\mathcal{M} \mid \dim_{\mathrm{H}}(J(p_c)) = 2\}$ *is not just dense, but residual.*

For the proof of lemma 5.11 and the claim in the last proof we introduce a new tool.

5.2 Holomorphic Motions

Definition 5.14. *Given* $X \subset \overline{\mathbb{C}}$ *and a complex manifold* Λ*, a holomorphic motion on* X *parametrized by* Λ *with base point* $\lambda_0 \in \Lambda$ *is a function* $i \colon \Lambda \times X \to \overline{\mathbb{C}}$*, which we denote by* $i_\lambda(z)$*, satisfying*

1. $i_{\lambda_0} \colon X \to \overline{\mathbb{C}}$ *is the identity map* id_X*,*
2. $\forall \lambda \in \Lambda \colon i_\lambda \colon X \to \overline{\mathbb{C}}$ *is injective,*
3. $\forall z \in X \colon i_{.}(z) \colon \Lambda \to \overline{\mathbb{C}}$ *is holomorphic.*

Note that there is no regularity assumption on i_λ. However, the λ-lemma from the last chapter shows that a strong regularity property is implicit in the definition. With our new notion we can reformulate the λ-lemma as follows.

Theorem 5.15 (λ-lemma)**.** *Suppose* $i \colon D \times X \to \overline{\mathbb{C}}$ *is a holomorphic motion on* $X \subset \overline{\mathbb{C}}$ *parametrized by the unit disk* D *with base point 0. Then there is an extension to a holomorphic motion* j *on* $\overline{X}$ *parametrized on* D*, which is jointly continuous and such that each* j_λ *is quasi-conformal.*

In fact, Sullivan and Thurston strengthened the λ-lemma in the following way, see [29, p. 244].

Theorem 5.16. *Take* i*,* X*, and* D *as in the* λ*-lemma. Then there exists an* $a > 0$ *such that* i *can be extended to a holomorphic motion on all of* $\mathbb{C}$ *parametrized by the disk of radius* a*. Most importantly, the constant* a *is universal, i.e. independent of* i *and* X*.*

By a slight abuse of notation we usually denote the extension by i as well. In this version, quasi-conformality is not included, but it follows just as in the λ-lemma. In addition, since the constant a is universal, the quasi-conformality statement can be strengthened alike.

Theorem 5.17. *With* a *denoting the universal constant, there exists a map* $K \colon [0, a) \to (0, \infty)$ *such that if* i *is an extension from the last theorem, then* i_λ *is* $K(|\lambda|)$*-quasi-conformal. Moreover,* $K(t) \to 1$ *as* $t \to 0$*.*

The proof follows the three similar ones in [17, p. 201-202], [29, p. 2] and [14, p. 96].

Proof. By the λ-lemma, i_λ is continuous. We will show that it is quasi-möbius from which quasi-conformality follows (see the appendix, theorem A.6). Fix a quadruple (z_1, z_2, z_3, z_4) of distinct points in $\mathbb{C}$ and set

$$f(\lambda) = \frac{i_\lambda(z_1) - i_\lambda(z_3)}{i_\lambda(z_1) - i_\lambda(z_4)} \cdot \frac{i_\lambda(z_2) - i_\lambda(z_4)}{i_\lambda(z_4) - i_\lambda(z_3)}$$

Note that $|f(\lambda)|$ is exactly the cross-ratio of the image points of (z_1, z_2, z_3, z_4) under i_λ, and $|f(0)|$ is the cross ratio of (z_1, z_2, z_3, z_4). Due to injectivity of i_λ the map f omits 0, 1 and ∞. Thus, the range of f is a three punctured sphere, which admits a Poincaré metric ρ. The Schwarz-Ahlfors-Pick theorem[11] (see [22, p. 3]) says that the Poincaré metric is non-increasing under hyperbolic maps, i.e.

$$\rho(f(\lambda), f(0)) \leq \rho_D(\lambda, 0)$$

where ρ_D denotes the Poincaré metric on the disk $D_a(0)$. Next, by completeness of ρ, if $\rho(z, w)$ is bounded and $|z| \to 0$, then also $|w| \to 0$. Hence, we may pick a family $\theta_r \colon (0, \infty) \to (0, \infty)$, $r > 0$, of continuous functions such that θ_r depends continuously on r, $\theta_r(t) \to 0$ as $t \to 0$ for each r, and $|z| \leq \theta_r(|w|)$ whenever $\rho(z, w) < r$. With $r(\lambda) = \rho_D(\lambda, 0)$ we get

$$|f(\lambda)| \leq \theta_{r(\lambda)}(CR(z_1, z_2, z_3, z_4))$$

where $CR()$ denotes the cross-ratio. This proves that the map i_λ is quasi-möbius and therefore, by theorem A.6, $\theta_{r_{(\lambda)}}(1)$-quasi-conformal for each λ. Furthermore, ρ, ρ_D, θ_r and $r(\lambda)$ are all independent of f and, hence, also independent of the motion. Lastly, choose K such that for each $\lambda \in D_a(0)$ we have $\theta_{r_{(\lambda)}}(1) \leq K(|\lambda|)$. $\square$

Let us briefly discuss the nature of the proofs to come. We show below that if i is an α-Hölder continuous map, then

$$\dim_{\mathrm{H}}(i(X)) \leq \frac{1}{\alpha} \cdot \dim_{\mathrm{H}}(X)$$

Moreover, we know from Mori's theorem (see the appendix, theorem A.6) that any K-quasi-conformal map is $\frac{1}{K}$-Hölder continuous. With the λ-lemma and Mori's theorem we can therefore establish a connection between $\dim_{\mathrm{H}}(X)$ and $\dim_{\mathrm{H}}(i_\lambda(X))$. In each of the proofs, the key is to find a suitable holomorphic motion such that X and $i_\lambda(X)$ are the sets in whose Hausdorff dimension we are interested in.

Proposition 5.18. *If $i \colon X \to \overline{\mathbb{C}}$ is Hölder continuous, then* $\dim_{\mathrm{H}}(i(X)) \leq \frac{1}{\alpha} \dim_{\mathrm{H}}(X)$.

Proof. Suppose i is Hölder continuous, i.e.

$$\exists C, \alpha > 0 \text{ such that } \forall x, y \in X \colon |i(x) - i(y)| < C|x - y|^\alpha$$

If $\mathcal{U}$ is a cover of X of sets of diameter at most δ, then $i(\mathcal{U})$ is a cover of $i(X)$ of sets of diameter at most $C\delta^\alpha$. Hence,

$$H_s^{C\delta^\alpha}(i(X)) \leq \inf_{\mathcal{U}} \sum_{U \in \mathcal{U}} \operatorname{diam}(i(U))^s \leq \inf_{\mathcal{U}} \sum_{U \in \mathcal{U}} C^s \operatorname{diam}(U)^{\alpha s} = C^s H_{\alpha s}^\delta(X)$$

By taking limits, we obtain $H_s(i(X)) \leq C^s H_{\alpha s}(X)$, which implies $\dim_{\mathrm{H}}(i(X)) \leq \frac{1}{\alpha} \dim_{\mathrm{H}}(X)$. $\square$

[11] Also known as generalized Schwarz lemma or Pick theorem.

One immediate consequence is the following useful result.

Corollary 5.19. *If* $v\colon X \to \overline{\mathbb{C}}$ *is bi-Lipschitz, then* $\dim_{\mathrm{H}}(v^{-1}(X)) = \dim_{\mathrm{H}}(X) = \dim_{\mathrm{H}}(v(X))$.

To finish the proof of theorem 5.5 we show a continuity result for the Hausdorff dimension. Suppose $f \in \{z^d + c \mid c \in \mathbb{C}\}$ and X_0 is a hyperbolic set for f. By the persistence of hyperbolic sets, each map g close to f has a hyperbolic set X_g.

Proposition 5.20. *The map* $g \to \dim_{\mathrm{H}}(X_g)$ *is continuous in a neighborhood of* f *in the set* $\{z^d + c \mid c \in \mathbb{C}\}$.

The proof given here is an expansion of the one in [24, p. 10].

Proof. Let U and h_g be as in the proposition on persistence of hyperbolic sets. Fix $g_0(z) = z^d + c_0 \in U$ and set $V = U \cap \{z^d + c \mid c \in \mathbb{C}\}$. Then V is a one-complex dimensional manifold, which can be embedded in $\mathbb{C}$, and by replacing V with a disk $D_r(g_0) \subset V$ and making an affine coordinate change, we may assume that V is the unit disk D and g_0 corresponds to the origin. For simplicity, we still write g for an element of D. This way,

$$i\colon D \times X \to \overline{\mathbb{C}},\ i_g(z) = h_g \circ h_{g_0}^{-1}(z)$$

is a holomorphic motion on $X = X_{g_0}$ parametrized by D with base point $0 \simeq g_0$. By theorem 5.17 each i_g has a $K(|g|)$-quasi-conformal extension. If we set $\alpha(g) = 1/K(|g|)$, then $X_g = i_g(X)$ is the image of X under an $\alpha(g)$-Hölder continuous map.
Now fix a point g_1 close to g_0 and let D_{g_1} denote the disk $\{g_1 + z \mid |z| < \epsilon\} \subset D$ with ϵ such that $g_0 \in D_{g_1}$. Let $w_{g_1}\colon D \to D_{g_1},\ w(z) = \epsilon z + g_1$ be the affine coordinate change of D_{g_1} to the unit disk. Then

$$j\colon D \times X_{g_1} \to \overline{\mathbb{C}},\ j_g(z) = h_{w_{g_1}(g)} \circ h_{g_1}^{-1}(z)$$

is a holomorphic motion on X_{g_1} parametrized by the unit disk with base point 0. By the same reasoning as before, $X = j_{w_{g_1}^{-1}(g_0)}(X_{g_1})$ is the image of X_{g_1} under an $\alpha(w_{g_1}^{-1}(g_0))$-Hölder continuous map. Here we use that the map K in theorem 5.17 is independent of the motion, but at the same time we needed to make the affine coordinate change to accommodate that the two motions have different base points g_0 and g_1. Recall our identification $0 \simeq g_0$ and observe that $w_{g_1}^{-1}(g_0) = \frac{-g_1}{\epsilon}$ is close to 0 whenever g_1 is. Since $\alpha(g) \to 1$ as $g \to 0$, given any $0 < \alpha < 1$, for all g_1 sufficiently close to $g_0 \simeq 0$ we have $\alpha < \alpha(g_1) < 1$ as well as $\alpha < \alpha(w_{g_1}^{-1}(g_0)) < 1$. By Hölder continuity

$$\alpha \cdot \dim_{\mathrm{H}}(X) \leq \dim_{\mathrm{H}}(X_g) \leq \frac{1}{\alpha} \cdot \dim_{\mathrm{H}}(X)$$

We can conclude continuity of $\dim_{\mathrm{H}}(X_g)$. □

Now, we can easily deduce the claim from the proof of the first main theorem.

Corollary 5.21 (Completion of the proof of theorem 5.6)**.** *The set* $\{c \in \partial\mathcal{M} \mid \dim_{\mathrm{hyp}}(p_c) > 2 - 1/n\}$ *is open in* $\partial\mathcal{M}$.

Proof. By definition

$$\dim_{\mathrm{hyp}}(f) = \sup\{\dim_{\mathrm{H}}(X) \mid X \text{ is a hyperbolic set for } f\}$$

and since $g \to \dim_{\mathrm{H}}(X_g)$ is continuous in $\{z^d + c \mid c \in \mathbb{C}\}$, the map $f \to \dim_{\mathrm{hyp}}(f)$ is lower semi-continuous in $\{z^d + c \mid c \in \mathbb{C}\}$. □

In order to prove lemma 5.11 we need two more technical results. The first is a generic statement about the Hausdorff dimension.

Lemma 5.22. *Given a compact subset $X \subset \overline{\mathbb{C}}$, there exists a point $z_0 \in X$ such that*

$$\lim_{r\to 0} \dim_{\mathrm{H}}(X \cap D_r(z_0)) = \dim_{\mathrm{H}}(X)$$

Proof. Suppose for contradiction it is not true, i.e.

$$\forall z \in X\ \exists \epsilon_z > 0\ \exists (n_{z,k})_{k\geq 0} \subset \mathbb{N}\ \forall k\colon\ \dim_{\mathrm{H}}(X \cap D_{\frac{1}{n_{z,k}}}(z)) < \dim_{\mathrm{H}}(X) - \epsilon_z$$

By compactness of X, we can pick points $z_1, \ldots, z_N$ such that

$$X = \bigcup_{1\leq j\leq N} X \cap D_{r_j}(z_j)$$

where r_j is short for $\frac{1}{n_{z_j,1}}$. It follows readily from the construction of the Hausdorff dimension that a union turns into a supremum.

$$\dim_{\mathrm{H}}(\bigcup_{1\leq j\leq n} X \cap D_{r_j}(z_j)) = \sup_{1\leq j\leq n} \dim_{\mathrm{H}}(X \cap D_{r_j}(z_j)) < \dim_{\mathrm{H}}(X) - \min_{1\leq j\leq N} \epsilon_{z_j} < \dim_{\mathrm{H}}(X)$$

This is a contradiction. □

This second result is the key to how we later link the Hausdorff dimension of $\partial\mathcal{M}$ to the one of some hyperbolic set.

Lemma 5.23. *Suppose $i_\lambda\colon X \to \overline{\mathbb{C}}$ is a holomorphic motion parametrized by the unit disk D with base point 0, and $v\colon D \to \overline{\mathbb{C}}$ is a holomorphic map with $v(0) = z_0 \in X$ and $v(\lambda) \not\equiv i_\lambda(z_0)$. Then*

$$\lim_{r\to 0} \dim_{\mathrm{H}}(X \cap D_r(z_0)) \leq \dim_{\mathrm{H}}(\{\lambda \in D \mid v(\lambda) \in i_\lambda(X)\}$$

This time, we will first need to work a bit to discover the holormophic motion that we want to apply the technique discussed earlier to. It will contain the information "$v(\lambda) \in i_\lambda(X)$". In order to construct this holomorphic motion we will rewrite "$v(\lambda) \in i_\lambda(X)$" as a root finding problem. We follow [24, p. 10-11] closely.

Proof. For simplicity we would like to have $z_0 = 0$ and $i_\lambda(0) = 0$ for all λ. Let us briefly check that we may assume this without loss of generality. We make a change of coordinates as follows. Let ϕ_λ be Möbius transformations depending analytically on λ such that $\phi_\lambda(i_\lambda(z_0)) = 0$ for all λ. Set $i'_\lambda(z) = \phi_\lambda \circ i_\lambda \circ \phi_0^{-1}(z)$ and $w(\lambda) = \phi_\lambda(v(\lambda))$. Then, $i'_\lambda(0) = 0$ and

$$w(\lambda) \in i'_\lambda(\phi_0(X)) \iff v(\lambda) \in i_\lambda(X)$$

Moreover,

$$\lim_{r\to 0} \dim_{\mathrm{H}}(\phi_0(X) \cap D_r(\phi_0(z_0))) = \lim_{r\to 0} \dim_{\mathrm{H}}(X \cap D_r(z_0))$$

as ϕ_0 is bi-lipschitz. From this change of coordinates it follows that it suffices to prove the lemma with the additional assumption that $z_0 = 0$ and $i_\lambda(0) = 0$ for all λ. We now drop the w and i' notation.
Suppose first that the derivative of v does not vanish at 0. We want to reduce the problem of finding λ

for which $v(\lambda) \in i_\lambda(X)$ to a root finding problem. Since $v'(0) \neq 0$ there exists $a > 0$ and $0 < \rho < 1$ such that on $D_\rho(0)$ v is injective and $|v(\lambda)| \geq a|\lambda|$. Define

$$b_r = \sup\{|i_\lambda(z)| \mid z \in X \cap D_r(0),\ |\lambda| \leq \rho\}$$

By the λ-lemma, i has a continuous extension and so $b_r \to 0$ as $r \to 0$. Thus, we can take r_0 with $b_r < a\rho$ for $0 < r < r_0$. Fix $0 < r < r_0$, $z \in X \cap D_r(0)$ and $\mu \in D_{\frac{a\rho}{b_r}}(0)$. Set $\Delta_\mu = \{\lambda \mid |\lambda| < \min(\rho, \frac{\rho}{|\mu|})\}$ and consider the root finding problem

$$v(\lambda) - i_{\lambda\mu}(z) = 0$$

in Δ_μ. On $\partial\Delta_\mu$ we have $|i_{\lambda\mu}(z)| \leq b_r$ by definition of b_r, and $b_r < |v(\lambda)|$ because $|v(\lambda)| \geq a|\lambda|$, $b_r < a\rho$ and $|\mu| < \frac{a\rho}{b_r}$. Since $v(0) = 0$ and v is injective on Δ_μ, this equation has a unique solution $\lambda(\mu, z)$ by Rouché's theorem[12]. Moreover, $\lambda(\mu, \cdot)$ is injective since $i_{\lambda\mu}$ is, and $\lambda(\mu, z)$ depends analytically on μ since $i_{\lambda\mu}(z)$ does.

Next, define

$$j\colon D \times v^{-1}(X \cap D_r(0)) \to \overline{\mathbb{C}},\ j_\mu(z) = \lambda(\frac{a\rho}{b_r}\mu, v(z))$$

This is a holomorphic motion on $v^{-1}(X \cap D_r(0))$ parametrized by the unit disk with base point 0. We check that it satisfies our needs: We have

$$v(\lambda(1, z)) = i_{\lambda(1,z)}(z) \in i_{\lambda(1,z)}(X)$$

and therefore

$$j_{\frac{b_r}{a\rho}}(v^{-1}(X \cap D_r(0))) = \{\lambda(1, v(z)) \mid z \in v^{-1}(X \cap D_r(0))\} \subset \{\lambda \in D \mid v(\lambda) \in i_\lambda(X)\}$$

As in the proof of proposition 5.20, each j_μ is $\alpha(\mu)$-Hölder continuous, where α is a function that tends to 1 as $\mu \to 0$. Also note that j_λ is a homeomoprhism onto its image as X is compact. Now we make extensive use that the quasi-conformal extension of j in the improved λ-lemma is defined on all of $\mathbb{C}$. Namely, these last two conditions allow us to apply theorem A.4, which tells us that also the inverse functions j_λ are quasi-conformal (even though this might not be a holomorphic motion itself). Moreover, the dilatation is the same. Hence, the inverse maps are also Hölder continuous with the same exponent, and we get

$$\alpha(\frac{b_r}{a\rho}) \dim_{\mathrm{H}}(v^{-1}(X \cap D_r(0))) \leq \dim_{\mathrm{H}}(\{\lambda \in D \mid v(\lambda) \in i_\lambda(X)\})$$

Lastly, observe that

$$\dim_{\mathrm{H}}(v^{-1}(X \cap D_r(0))) = \dim_{\mathrm{H}}(X \cap D_r(0))$$

as v is bi-Lipschitz and let $r \to 0$.

Suppose now $v'(0) = 0$ and that this root has order m. After the coordinate change done in the beginning, we have $v \not\equiv 0$ by hypothesis. Moreover, in that coordinate change we could as well have taken Möbius transformations so that we may assume $\infty \in X$ and $i_\lambda(\infty) = \infty$. Define $G(z) = z^m$ and let $w\colon D \to \overline{\mathbb{C}}$

[12] Rouché's theorem states that if f and g are two functions that are holomorphic in K, where K is a bounded set with continuous boundary, and satisfy $|g| < |f|$ on ∂K, then f and $f + g$ have the same number of roots inside K. Here we apply it with $f = v$ and $g = -i_{\cdot\mu}(z)$.

and $i'_\lambda\colon G^{-1}(X) \to \overline{\mathbb{C}}$ denote lifts of v and i_λ with respect to G, i.e. $v = G \circ w$ and $i_\lambda \circ G = G \circ i'_\lambda$ for all λ in D. Applying the first step to w and i' we get

$$\lim_{r\to 0} \dim_{\mathrm{H}}(G^{-1}(X) \cap D_r(0)) \leq \dim_{\mathrm{H}}(\{\lambda \in D \mid w(\lambda) \in i'_\lambda(G^{-1}(X))\}$$

We conclude using

$$w(\lambda) \in i'_\lambda(G^{-1}(X)) \iff v(\lambda) \in i_\lambda(X)$$

and that G is bi-Lipschitz. □

Finally, we can prove lemma 5.11. Recall that we want to show the inequality

$$\dim_{\mathrm{hyp}}(f_{c_0}) \leq \dim_{\mathrm{H}}(U \cap \partial\mathcal{M})$$

We slightly modify the proof in [24, p. 11-12] with an addition from [16, p. 238]. The idea is to construct a function v such that the condition "$v(\lambda) \in i_\lambda(X)$" in the last lemma actually reads something like "$c \in U \cap \partial\mathcal{M}$". The assertion then follows easily.

Proof of Lemma 5.11. By definition, given any $\epsilon > 0$ there is a hyperbolic set X for $f = f_{c_0}$ with

$$\dim_{\mathrm{H}}(X) > \dim_{\mathrm{hyp}}(f) - \epsilon$$

By the persistence of hyperbolic sets, there is a neighborhood $U' \subset U$ of c_0, there are hyperbolic sets X_c for f_c, and a holomorphic motion $h\colon U' \times X \to \overline{\mathbb{C}}$ on X with $h_c \circ f = f_c \circ h_c$. Let z_0 be the point given by lemma 5.22. By proposition 5.20 we may assume that

$$\forall c \in U'\colon \ \lim_{r\to 0} \dim_{\mathrm{H}}(X_c \cap D_r(h_c(z_0))) > \dim_{\mathrm{H}}(X) - \epsilon$$

after shrinking U' if necessary. To apply the previous lemma we need to find a suitable function v. Lemma III.2 in [17, p. 204] yields a $c_1 \in U'$, an $N > 0$, and a critical point ω of f_{c_1} such that $f_{c_1}^N(\omega) = h_{c_1}(z_0) \in X_{c_1}$[13]. Since every f_c is of the form $z^d + c$, this critical point is, in fact, the origin (∞, being a fixed point, is excluded since hyperbolic sets obviously cannot contain critical points).

To apply the above lemma we need to make an affine coordinate change from U' to the unit disk. Take a small $s > 0$ such that $U'' = \{s\lambda + c_1 \mid \lambda \in D\} \subset U'$ and set

$$v(\lambda) = f_{s\lambda+c_1}^N(0)\ ,\ i_\lambda(z) = h_{s\lambda+c_1} \circ h_{c_1}^{-1}(z)$$

Note that the condition $v(\lambda) = i_\lambda(h_{c_1}(z_0))$ would imply that $\mathrm{Orb}_{f_{s\lambda+c_1}}(0)$ eventually enters the Julia set of $f_{s\lambda+c_1}$ as of proposition 5.9. Since U'' intersects $\partial\mathcal{M}$[14], there are points for which this is not the case (for example, we can employ proposition 3.13). Hence, the requirement $v(\lambda) \not\equiv i_\lambda(h_{c_1}(z_0))$ is fulfilled. The lemma yields

$$\dim_{\mathrm{hyp}}(f) - 2\epsilon < \lim_{r\to 0} \dim_{\mathrm{H}}(X_{c_1} \cap D_r(h_{c_1}(z_0))) \leq$$
$$\dim_{\mathrm{H}}(\{\lambda \in D \mid v(\lambda) \in i_\lambda(X_{c_1})\}) \leq \dim_{\mathrm{H}}(\{c \in U'' \mid f_c^N(0) \in X_c\})$$

[13] In the notation of lemma III.2 in [17, p. 204] we use $W_0 = U'$ and $\phi(w) = h_w(z_0)$. Then

$$f_w^N(\phi(w)) = h_w(f_{c_0}^N(z_0))$$

shows that if one $\phi(w)$ is periodic, then all $\phi(w)$ are periodic under f_w, i.e. the hypothesis of the lemma are satisfied. However, since U' meets $\partial\mathcal{M}$ and no point in $U' \cap \partial\mathcal{M}$ is persistently non-hyperbolic as of theorem B in [17, p. 199], $H(f)$ cannot contain all of W_0, i.e. lemma III.2 cannot hold. Thus, the hypothesis of lemma III.1 in [17, p. 204] must be violated. This yields the c_1, ω and N as desired.

To complete the proof, we show that $\{c \in U'' \mid f_c^N(0) \in X_c\}$ is a subset of $U'' \cap \partial\mathcal{M}$. Let $c_2 \in \{c \in U'' \mid f_c^N(0) \in X_c\}$. Note that if $\mathrm{Orb}_{f_{c_2}}(0)$ enters the Julia set of f_{c_2}, then $c_2 \in \partial\mathcal{M}$ or c_2 is in a non-hyperbolic component. Suppose for contradiction it was in a non-hyperbolic component. Then the critical value 0 lies in the Julia set by a similar argument as before, involving the Sullivan classification. It is shown in [17, p. 199] that for parameters in $\mathbb{C} \setminus \partial\mathcal{M}$, the Julia set moves continuously[15] with the parameter. In particular, for any compact neighborhood K of c_2 in its non-hyperbolic component, $f_c^n(0)$ remains bounded away from ∞ for any $c \in K$. Thus, the proof of the lemma follows from the following claim: $f_{c_2}^N(0) \in X_{c_2}$ implies that $(f_{c_2}^n(0))_{\{c \text{ close to } c_2\}}$ is not a normal family.
Let $r > 0$ be so that the disk around c_2 of radius r is contained inside U''. For notational convenience, define $z_1 = f_{c_2}^N(0) \in X_{c_2}$, $z_2 = h_{c_2}^{-1}(z_1) \in X$ and also new functions

$$a(\lambda) = f_{r\lambda+c_2}^N(0) \quad \text{and} \quad b(\lambda) = h_{r\lambda+c_2}(z_2)$$

Note that $a(0) = b(0)$. Furthermore, we claim that $a(\lambda) \not\equiv b(\lambda)$ on D. Suppose this was not true. As $D_r(c_2)$ is a disk inside U'', the identity principle tells us that $f_{s\lambda+c_1}^N(0) \equiv h_{s\lambda+c_1}(z')$ on U'' for some z'. But since $f_{c_1}^N(0) = h_{c_1}(z_0)$ and h_{c_1} is injective, we must have $z' = z_0$, which contradicts $v(\lambda) \not\equiv h_{s\lambda+c_1}(z_0)$. Therefore, $a(\lambda) \not\equiv b(\lambda)$ on D. Let p be the order of the zero of $a(\lambda) - b(\lambda)$ at 0 so that

$$a(\lambda) - b(\lambda) = t\lambda^p + \mathcal{O}(\lambda^{p+1})$$

for some $t \in \mathbb{C}$. We will prove the following statement by induction:

$$f_{r\lambda+c_2}^n(a(\lambda)) - f_{r\lambda+c_2}^n(b(\lambda)) = (f_{c_2}^n)'(z_1) \cdot t\lambda^p + \mathcal{O}(\lambda^{p+1})$$

To show the induction step, note that expanding $f_\lambda(z)$ into its power series with respect to z and λ at the same time yields

$$\begin{aligned} f_\lambda(z) = f_{\lambda_0}(z') &+ \Big(\frac{\partial}{\partial\lambda}\Big|_{\lambda=\lambda_0} f_\lambda\Big)(z') \cdot (\lambda - \lambda_0) + f'_{\lambda_0}(z') \cdot (z - z') \\ &+ \Big(\frac{\partial}{\partial\lambda}\Big|_{\lambda=\lambda_0} f'_\lambda\Big)(z') \cdot (z - z') \cdot (\lambda - \lambda_0) + O(z - z')^2 + O(\lambda - \lambda_0)^2 \end{aligned}$$

In our case

$$\Big(\frac{\partial}{\partial\lambda}\Big|_{\lambda=0} f_{r\lambda+c_2}\Big)(z) = r \quad \text{and} \quad \Big(\frac{\partial}{\partial\lambda}\Big|_{\lambda=0} f'_{r\lambda+c_2}\Big)(z) = 0$$

Expanding around $z_n = f_{c_2}^n(z_1)$ we get

$$f_{r\lambda+c_2}\big(f_{r\lambda+c_2}^n(a(\lambda))\big) - f_{r\lambda+c_2}\big(f_{r\lambda+c_2}^n(b(\lambda))\big) = f'_{c_2}(z_n) \cdot \Big(f_{r\lambda+c_2}^n(a(\lambda)) - f_{r\lambda+c_2}^n(b(\lambda))\Big)$$

Using the induction hypothesis together with the fact

$$(f^{n+1})'(z) = f'(f^n(z)) \cdot (f^n)'(z)$$

finishes the induction step. Unraveling the definition of $b(\lambda)$ and $c(\lambda)$, we have just shown that

$$f_{r\lambda+c_2}^{n+N}(0) - h_{r\lambda+c_2} \circ f^n(z_2) = (f_{c_2}^n)'(z_1) \cdot t\lambda^p + \mathcal{O}(\lambda^{p+1})$$

[14] Technically, we should have been more careful to ensure that U'' does indeed intersect $\partial\mathcal{M}$. However, this is done easily by first shrinking U' a lot, picking c_1 in the shrunken set, and then taking a disk around c_1 in the original U'.

[15] In the Hausdorff metric.

Now we take the derivative $\frac{\partial^p}{\partial \lambda^p}$ on both sides. Clearly, the term

$$\frac{\partial^p}{\partial \lambda^p} h_{r\lambda + c_2} \circ f^n(z_2)$$

is bounded in n close to $\lambda = 0$ since its maximal value only depends on $h_{r\lambda+c_2}$. The right hand side

$$\frac{\partial^p}{\partial \lambda^p} \Big((f_{c_2}^n)'(z_1) \cdot t\lambda^p + \mathcal{O}(\lambda^{p+1}) \Big) = (f_{c_2}^n)'(z_1) \cdot p!t + \mathcal{O}(\lambda)$$

diverges to ∞ because $z_1 \in X_{c_2}$ and by definition of a hyperbolic set $|(f_{c_2}^n)'(z_1)| \to \infty$ as $n \to \infty$. Therefore, $f_{r\lambda+c_2}^{n+N}(0)$ is unbounded close to $\lambda = 0$, hence not a normal family. $\square$

Now that lemma 5.11 is shown, the proof of the two fractal theorems is complete. We have successfully justified calling the Mandelbrot set a fractal.

A Preliminaries on Quasi-conformal Maps

For quasi-conformality we use the definition given by Mañé, Sad and Sullivan in [17, p. 199].

Definition A.1. *A map $f\colon X \to Y$, where X and Y denote two subsets of $\overline{\mathbb{C}}$, is K-quasi-conformal if*

$$\sup_{x\in X} \limsup_{t\to 0} \frac{\sup_{y\in\partial B_t(x)} d(f(x), f(y))}{\inf_{z\in\partial B_t(x)} d(f(x), f(z))} \leq K$$

Furthermore, we require a K-quasi-conformal map to be continuous and injective. A map is quasi-conformal if it is K-quasi-conformal for some $K < \infty$. The number K is called dilatation of f.

In the chapter on the Hausdorff dimension of $\partial\mathcal{M}$ we use a connection between quasi-conformality and Hölder continuity. It is proved in [1, p. 30] under the name of Mori's theorem.

Theorem A.2. *A K-quasi-conformal map is $\frac{1}{K}$-Hölder continuous.*

This corollary we use several times in the last chapter.

Corollary A.3. *Where a holomorphic map has non zero derivative it is bi-Lipschitz.*

Proof. A holomorphic map is conformal, i.e. 1-quasi-conformal, when its derivative does not vanish. Moreover, whenever its derivative does not vanish it has a holomorphic inverse. □

Under sufficient conditions, quasi-conformality of a map is enough for quasi-conformality of its inverse, see [12, p. 5].

Theorem A.4. *If f is defined on a domain and is a homeomorphism onto its image, then the inverse is also quasi-conformal and has the same dilatation as f.*

Sometimes it is convenient to check a map for a stronger notion than quasi-conformality. In [30], so called quasi-möbius maps are introduced. In the original paper, these maps are considered on arbitrary, one-point extended, metric spaces, but we will restrict ourselves to the Riemann sphere.

Definition A.5. *Suppose $A \subset \overline{\mathbb{C}}$ is a subset, $\theta\colon [0,\infty) \to [0,\infty)$ is a homeomorphism and $f\colon A \to \overline{\mathbb{C}}$ is continuous and injective. We say f is θ-quasi-möbius if for any quadruple (z_1, z_2, z_3, z_4) of distinct points in A*

$$CR(f(z_1), f(z_2), f(z_3), f(z_4)) \leq \theta(CR(z_1, z_2, z_3, z_4))$$

where $CR()$ denotes the cross-ratio of four points. Naturally, f is said to be quasi-möbius if there exists some homeomorphism θ such that f is θ-quasi-möbius.

It is proved in [30, p. 231] that this notion is indeed stronger than quasi-conformality.

Theorem A.6. *Suppose $A \subset \overline{\mathbb{C}}$ and $f\colon A \to \overline{\mathbb{C}}$ is θ-quasi-möbius. Then f is quasi-conformal and the dilatation of f is bounded by $\theta(1)$.*

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